

Studies of perturbed three vortex dynamics

Denis Blackmore

Department of Mathematical Sciences, New Jersey Institute of Technology, Newark, NJ 07102-1982

Lu Ting

Courant Institute of Mathematical Sciences, New York University, New York, NY 10012-1186

Omar Knio

Department of Mechanical Engineering, The Johns Hopkins University, Baltimore, MD 21218

It is well known that the dynamics of three point vortices moving in an ideal fluid in the plane can be expressed in Hamiltonian form, where the resulting equations of motion are completely integrable in the sense of Liouville and Arnold. The focus of this investigation is on the persistence of regular behavior (especially periodic motion) associated to completely integrable systems for certain (admissible) kinds of Hamiltonian perturbations of the three vortex system in a plane. After a brief survey of the dynamics of the integrable planar three vortex system, it is shown that the admissible class of perturbed systems is broad enough to include three vortices in a half-plane, three coaxial slender vortex rings in three-space, and ‘restricted’ four vortex dynamics in a plane. Included are two basic categories of results for admissible perturbations: (i) general theorems for the persistence of invariant tori and periodic orbits using Kolmogorov-Arnold-Moser and Poincaré-Birkhoff type arguments; and (ii) more specific and quantitative conclusions of a classical perturbation theory nature guaranteeing the existence of periodic orbits of the perturbed system close to cycles of the unperturbed system, which occur in abundance near centers. In addition, several numerical simulations are provided to illustrate the validity of the theorems as well as indicating their limitations as manifested by transitions to chaotic dynamics.

Physics and Astronomy Classification Scheme: 47.10.Df, 47.10.Fg, 47.15.ki, 47.20.Ky, 47.32.C-, 47.32.cb, 47.52.+j

I. INTRODUCTION

Although virtually all research on vortex dominated fluid flows has its roots in the seminal work of Helmholtz [29] (*cf.* [38]), specific advances in the dynamics of point vortices moving in an *ideal* (= inviscid, incompressible) fluid in a planar region, which we shall refer to as *n vortex dynamics* or the *n vortex problem*, can be traced back to the pioneering work of Kirchhoff [33] and Gröbli [28]. Kirchhoff was the first to describe the Hamiltonian structure of the *n* vortex problem, which he used to derive some fundamental integrals of the motion, while Gröbli conducted a detailed analysis of the three vortex problem that included what was essentially a proof of the integrability by quadratures of the problem without employing Hamiltonian formalism, although he did not give a complete description of the dynamics. About twenty years after Gröbli’s remarkable work, Goryachev [27] took up the problem again, and was able to obtain new insights concerning the dynamics of point vortices. Shortly thereafter, Poincaré [47] put his own imprimatur on vortex dynamics, just as he did in so many other fields of research.

Some seventy years after the pioneering work of Kirchhoff and Gröbli, Synge [52] was able - using trilinear coordinates - to fill many of the gaps in the dynamical picture of the three vortex problem left by earlier studies. Among Synge’s most important contributions were the derivation of integrals and the characterization of the critical points in terms of trilinear coordinates, and the identification of a single parameter - involving the sum of the product of the vortex strengths - that distinguishes three distinct types of qualitative dynamics (the elliptic, hyperbolic, and parabolic types) depending on whether this parameter is positive, negative or zero. Twenty years later, Aref [1, 2] used the Hamiltonian based investigations of Novikov [45] to rediscover the advantages of studying the three vortex problem in the context of trilinear coordinates. In the process, Aref identified additional properties of three vortex dynamics, and initiated research on chaos in point vortex dynamics and its relation to turbulent flows.

Subsequently, Tavantzis & Ting [53], taking Synge’s approach as their point of departure, derived a new constant of motion in trilinear coordinates, which they used together with some classical perturbation

techniques to nearly complete the characterization of three point vortex dynamics. Using linear analysis, they determined the stability of all isolated stationary points in the trilinear plane, and showed that expanding and contracting configurations are, respectively, stable and unstable, which implies that contracting similarity solutions and the eventual collision of three vortices is unstable. More recently, Ting *et al.* [56] employed techniques from nonlinear stability analysis to supply the few missing details in the dynamics; in particular, they showed that orbits starting just off the contracting configuration branch of the singular curve of critical points in the parabolic case are ultimately attracted to the expanding branch of this curve.

The Hamiltonian approach introduced by Kirchhoff [33], further developed by Lin [41], and perfected for point vortex problems by Novikov [45] has proven to be very useful in vortex dynamics research. Although not essential for solving the three vortex problem - as demonstrated in [28], [52], [53], and [56] - presumably one could use the integrals in involution to reduce it to a solvable one-degree-of-freedom Hamiltonian system along the lines indicated by Borisov and his collaborators in such papers as [15], [16], and [17]. On the other hand, the symplectic structure underlying Hamiltonian dynamics has proven to be extraordinarily effective in resolving a wide variety of vortex problems such as the formulation of point vortex dynamics on the sphere by Bogomolov [14], the proof of complete integrability of the three vortex problem on the sphere by Kidambi & Newton [34] (see also [15], [17], [35] and [44]), verification of non-integrability of the general n vortex problem and $(n - 1)$ coaxial vortex ring problem for $n > 3$ by Bagrets & Bagrets [6] (see also Ziglin [58]), and several other important results such as in [2], [3], [23], [37], [40], [49], and [57].

It is in the Hamiltonian perturbation of integrable point vortex dynamics where the Hamiltonian approach has proven to be particularly useful. This is largely due to the availability of two of the most important results in finite-dimensional symplectic dynamics: Kolmogorov-Arnold-Moser (KAM) theory (such as in [4], [5], [31], and [42]) and Poincaré-Birkhoff (PB) theory and its extensions and variants (see *e.g.* [7], [10], [11], [19], [21], [22], [24], [25], [26], [30], [31], [43], [46], and [48]). Examples of such applications manifold: Khanin [32] used a KAM theory inspired method to show that there exist subsets of initial configurations for the (non-integrable) four vortex problem in the plane leading to regular (integrable like) motion; namely, quasiperiodic orbits on (deformed) invariant (KAM) tori (*cf.* Celletti & Falcolini [20]). A KAM theory based argument combined with a deft application of Jacobi canonical transformations enabled Lim [39] to prove the existence of quasiperiodic flow regimes in lattice vortex systems. Blackmore & Knio [8], using the fact that slender coaxial vortex ring dynamics can be viewed as a perturbation of point vortex motion, employed an innovative KAM theory type result to prove the persistence of KAM tori and periodic orbits for an ample set of initial positions of three rings sufficiently close to one another when the rings having vortex strengths of the same sign. Later, Blackmore *et al.* [11], employing an analogous KAM theory approach in concert with a novel extension of PB theory, generalized this to any finite number of coaxial vortex rings with strengths of the same sign. Similar results were obtained by Blackmore & Champanerkar [13] using the same type of approach for any finite number of point vortices in a plane or half-plane. In addition, Ting & Blackmore [55] and Blackmore *et al.* [12] employed analogous ideas, together with trilinear coordinates, to prove the persistence of regular flow regimes for three coaxial vortex rings and three vortices in a half-plane, respectively, when the vortex strengths are of differing signs.

In this paper we shall, after a brief description of three vortex dynamics, assemble and extend some of our recent results on the persistence of quasiperiodic flows on KAM tori and periodic orbits for certain types of non-integrable Hamiltonian perturbations of three vortex dynamics. Although we shall include some interesting - fundamentally qualitative - results on the existence of quasiperiodic and periodic orbits for perturbations of three vortex dynamics, we intend to focus our attention on more quantitative behavior concerning the persistence of periodic orbits that are, in an appropriate sense, near periodic orbits of the unperturbed completely integrable system. We shall obtain our results using a judicious combination of Hamiltonian techniques and more classical perturbation methods that are closely linked with the representation of the unperturbed dynamics in trilinear coordinates.

The equations of motion for three vortex dynamics in both Hamiltonian form and planar trilinear coordinates are presented in Section II, along with a brief review of the trilinear phase portrait that emphasizes those dynamical features that exploited extensively in the sequel. Next, in Section III, we describe specific types of perturbations that are subsumed by the Hamiltonian perturbations for which our main conclusions are proven in subsequent sections. The specific perturbations considered are three vortices in a half-plane, a ‘restricted’ four vortex problem in the plane, and three coaxial vortex rings. We prove qualitative theorems in Section IV on the persistence of quasiperiodic flows on KAM tori interspersed with periodic orbits for

Hamiltonian perturbations of three vortex dynamics subject to hypotheses satisfied by the examples in the preceding section. A more classical perturbation approach is used in Section V to show that the perturbations of type introduced in Section IV have the property of having periodic orbits that are close - in an appropriate coordinate system - to certain cycles of the three vortex system.

In the penultimate part of the paper, Section VI, we present a variety of numerical simulations that illustrate the persistence of regularity associated with the three vortex problem under perturbations satisfying the properties in our main results. The properties in question provide sufficient conditions on the initial configurations of point vortices or slender coaxial vortex rings for the existence of invariant tori and periodic orbits, and for periodic orbits of the perturbed system that are close to those of the unperturbed system. A number of simulations are also included to show how regularity breaks down - signalled by the emergence of chaos - as the limitations imposed by the hypotheses in our theorems are exceeded. Finally, in Section VII, we summarize and distill the most important conclusions in the paper, and indicate some promising directions for related future research.

II. UNPERTURBED DYNAMICS

In this section we describe the equations of motion of point vortices in an ideal fluid in the plane, and give a rather complete characterization of the dynamics for the integrable three vortex system. Knowing the dynamics in the three vortex case shall prove quite useful in our subsequent analysis and description of periodic orbits for perturbed systems that are perturbations of cycles of the unperturbed system.

A. Governing equations

We begin with the general problem of n point vortices of respective nonzero strengths $\Gamma_1, \dots, \Gamma_n$ moving in an ideal fluid in the (complex) plane \mathbb{C} ($= \mathbb{R}^2$) and located at the positions $z_k = x_k + iy_k$, $1 \leq k \leq n$, respectively. The equations of motion in complex form are

$$\dot{\bar{z}}_k = i \sum_{j=1, j \neq k}^n \kappa_j (z_j - z_k)^{-1}, \quad (1 \leq k \leq n) \quad (1)$$

where the overbar indicates the complex conjugate, the overdot denotes differentiation with respect to t , and $\kappa_j := \Gamma_j/2\pi$, $1 \leq j \leq n$, which can be recast as the complex Hamiltonian equation

$$\kappa * \dot{\mathbf{z}} = 2i\partial_{\mathbf{z}} H_0, \quad (2)$$

where $\kappa := (\kappa_1, \dots, \kappa_n)$, $\mathbf{z} := (z_1, \dots, z_n)$, $*$ denotes the usual (Cartesian product ring) product in \mathbb{C}^n , $\partial_{\mathbf{z}} H_0 := (\partial_{z_1} H_0, \dots, \partial_{z_n} H_0)$, and the Hamiltonian function is given as

$$H_0 := - \sum_{1 \leq j < k \leq n} \kappa_j \kappa_k \log |z_j - z_k|. \quad (3)$$

We note here that to guarantee smoothness of the system (1), the phase space needs to be defined as

$$\mathbb{C}_{\#}^n := \{\mathbf{z} \in \mathbb{C}^n : z_j \neq z_k \ \forall j \neq k\},$$

and we introduce the following notation that will prove useful in the sequel:

$$\mathfrak{K}_n^{(1)} := \sum_{j=1}^n \kappa_j, \quad \mathfrak{K}_n^{(2)} := \sum_{1 \leq j < k \leq n} \kappa_j \kappa_k.$$

The governing equations can be converted into the more familiar real Hamiltonian form in \mathbb{R}^{2n} (or more properly $\mathbb{R}_{\#}^{2n}$)

$$\dot{x}_k = \kappa_k^{-1} \partial_{y_k} H_0 = \{H_0, x_k\}, \quad \dot{y}_k = -\kappa_k^{-1} \partial_{x_k} H_0 = \{H_0, y_k\}, \quad (1 \leq k \leq n) \quad (4)$$

where $\mathbb{R}_{\#}^{2n}$ is $\mathbb{C}_{\#}^n$ in the usual representation in real coordinates, and the (nonstandard) Poisson bracket is defined as

$$\{f, g\} := \sum_{k=1}^n \kappa_k^{-1} \left(\frac{\partial f}{\partial y_k} \frac{\partial g}{\partial x_k} - \frac{\partial f}{\partial x_k} \frac{\partial g}{\partial y_k} \right).$$

It is easy to see that

$$H_0, J := \sum_{k=1}^n \kappa_k z_k, K := \sum_{k=1}^n \kappa_k |z_k|^2, \quad (5)$$

representing the total energy, the linear momentum (impulse) and the angular momentum, respectively, are integrals of the system (1) (= (2) = (4)). From these we can construct the following three functionally independent, real constants of motion that are in involution:

$$H_0, K, L := (\Re J)^2 + (\Im J)^2, \quad (6)$$

where \Re and \Im denote the real and imaginary part, respectively, of a complex number, and it is easy to verify the involutivity of these integrals by computing that

$$\{H_0, K\} = \{H_0, L\} = \{K, L\} = 0. \quad (7)$$

Accordingly the dynamics of three point vortices in a plane is completely integrable in the sense of Liouville and Arnold, which we shall denote as *LA-integrable*. Whence it follows from Liouville-Arnold theory and the Poincaré-Birkhoff fixed point theorem and its extensions (see *e.g.* [5], [7], [10], [13], [19], [21], [22], [24], [25], [26], [30], [31], [44], and [46]) that three vortex dynamics is quasiperiodic, confined to invariant 3-tori and exhibits periodic orbits of arbitrarily large periods for all combinations of nonvanishing vortex strengths.

B. Dynamics in trilinear coordinates

Three vortex dynamics, which we summarize here, can be described in a very efficient manner using trilinear coordinates, which have their roots in algebraic geometry.. Our approach follows that of Synge [52], Tavantzis & Ting [53], and Ting *et al.* [56], and we refer the reader to these sources and Aref [1, 2] for further details. We begin with a dynamic formulation introduced by Gröbli [28] in terms of the sides of the (possibly degenerate) triangular configuration denoted as $R_1 := |z_2 - z_3|$, $R_2 := |z_1 - z_3|$, and $R_3 := |z_1 - z_2|$, which can be expressed as

$$\begin{aligned} R_1 \dot{R}_1 &= 2A\kappa_1 (R_2^{-1} - R_3^{-1}), \\ R_2 \dot{R}_2 &= 2A\kappa_2 (R_3^{-1} - R_1^{-1}), \\ R_3 \dot{R}_3 &= 2A\kappa_3 (R_1^{-1} - R_2^{-1}), \end{aligned} \quad (8)$$

where A denotes the oriented area of the configuration defined to be positive (negative) for a counterclockwise (clockwise) arrangement of the vertices ordered as z_1, z_2, z_3 . Naturally the points $\mathbf{R} = (R_1, R_2, R_3)$ in Euclidean three-space that comprise the domain of (8) must be confined to the first octant (and also avoid or compensate for singularities on the bounding coordinate planes). It is interesting to note that the stationary (fixed) points of (8) correspond to equilateral configurations, or collinear configurations satisfying the additional condition $\dot{A} = 0$, where the extra condition in the collinear case is necessitated by the singular nature of the vector field at such points. Observe that the systems (1), (4) and (8) are invariant under the transformation $t \rightarrow -t, \kappa_j \rightarrow -\kappa_j$ ($1 \leq j \leq 3$), hence we may assume without loss of generality that

$$\kappa_1 \geq \kappa_2 > 0 \text{ and } \kappa_2 \geq \kappa_3, \quad (9)$$

which we shall do throughout the sequel.

The triangle inequality imposes additional restrictions on the domain of (8). We shall define this restricted domain in accordance with the approach of Synge [52] (and Tavantzis & Ting [53]), wherein one can find some excellent figures representing the concepts described below, by first introducing the equilateral triangle $\mathcal{T} := P_1 P_2 P_3$ of height one determined by intersecting the first octant with the plane \mathcal{P} defined as the locus of $R_1 + R_2 + R_3 = \sqrt{2/3}$, where $P_1 := (\sqrt{2/3}, 0, 0)$, $P_2 := (0, \sqrt{2/3}, 0)$, and $P_3 := (0, 0, \sqrt{2/3})$. Observe that the first octant can be viewed as the fiber bundle over \mathcal{T} with the rays emanating from the origin (but not including the origin) as fibers. It is easy to verify from the triangle inequality that the proper base space of the bundle defining the admissible points \mathcal{D} for (8) is the subtriangle $T = Q_1 Q_2 Q_3$, where Q_1 is the midpoint of the edge $P_2 P_3$, Q_2 is the midpoint of the edge $P_1 P_3$, and Q_3 is the midpoint of the edge $P_1 P_2$,

so that $Q_1 := (0, \sqrt{1/6}, \sqrt{1/6})$, $Q_2 := (\sqrt{1/6}, 0, \sqrt{1/6})$, and $Q_3 := (\sqrt{1/6}, \sqrt{1/6}, 0)$. We note that the bundle projection $\rho : \mathcal{D} \rightarrow T$ is just the radial projection on T .

Points in the triangle T can be assigned *trilinear coordinates* $\mathbf{x} = (x_1, x_2, x_3)$ defined as

$$x_j := R_j (R_1 + R_2 + R_3)^{-1}, \quad (1 \leq j \leq 3) \quad (10)$$

which we note are not independent inasmuch as they satisfy $x_1 + x_2 + x_3 = 1$, and represent distance from the edges of T ; in particular, x_1 , x_2 , and x_3 are, respectively, the distances of \mathbf{x} from the edges P_2P_3 , P_1P_3 , and P_1P_2 . The points in the planar triangle T can also be conveniently represented in terms of the following coordinates:

$$\alpha := (1/\sqrt{3})(x_2 - x_1), \quad \beta := x_3, \quad (11)$$

with ‘inverse’ transformation

$$x_1 = (1/2)(1 - \beta - \alpha\sqrt{3}), \quad x_2 = (1/2)(1 - \beta + \alpha\sqrt{3}), \quad x_3 = \beta. \quad (12)$$

The system (8) projected on T assumes the following form in trilinear coordinates

$$\begin{aligned} \dot{x}_1 &= \Lambda \left\{ \kappa_1 x_1 (x_3^2 - x_2^2) - x_1 [\kappa_1 x_1 (x_3^2 - x_2^2) + \kappa_2 x_2 (x_1^2 - x_3^2) + \kappa_3 x_3 (x_2^2 - x_1^2)] \right\}, \\ \dot{x}_2 &= \Lambda \left\{ \kappa_2 x_2 (x_1^2 - x_3^2) - x_2 [\kappa_1 x_1 (x_3^2 - x_2^2) + \kappa_2 x_2 (x_1^2 - x_3^2) + \kappa_3 x_3 (x_2^2 - x_1^2)] \right\}, \\ \dot{x}_3 &= \Lambda \left\{ \kappa_3 x_3 (x_2^2 - x_1^2) - x_3 [\kappa_1 x_1 (x_3^2 - x_2^2) + \kappa_2 x_2 (x_1^2 - x_3^2) + \kappa_3 x_3 (x_2^2 - x_1^2)] \right\}, \end{aligned} \quad (13)$$

where

$$\Lambda := 2A \left[(R_1 R_2 R_3)^{-1} (R_1 + R_2 + R_3) \right]^2. \quad (14)$$

The dimensionality of (12) can be reduced by one by recasting the system in α, β - coordinates as

$$\begin{aligned} \dot{\alpha} &= \Lambda \left\{ \left(1/8\sqrt{3} \right) \left[\kappa_2 (1 - 3\beta - \alpha\sqrt{3}) (1 - (\beta - \alpha\sqrt{3})^2) + \right. \right. \\ &\quad \left. \left. \kappa_1 (1 - 3\beta + \alpha\sqrt{3}) (1 - (\beta + \alpha\sqrt{3})^2) \right] - \alpha \Xi \right\}, \\ \dot{\beta} &= \Lambda \left\{ \kappa_3 \sqrt{3} \alpha \beta (1 - \beta) - (\beta/8) \Xi \right\}, \end{aligned} \quad (15)$$

where

$$\begin{aligned} \Xi &:= \left\{ -\kappa_1 (1 - 3\beta + \alpha\sqrt{3}) \left[1 - (\beta + \alpha\sqrt{3})^2 \right] + \right. \\ &\quad \left. \kappa_2 (1 - 3\beta - \alpha\sqrt{3}) \left[1 - (\beta - \alpha\sqrt{3})^2 \right] + \kappa_3 8\sqrt{3} \alpha \beta (1 - \beta) \right\}, \end{aligned} \quad (16)$$

and the factor Λ can be absorbed into a rescaling of time in order to simplify the description of the phase portrait on T .

The system (13) or (15) actually includes two possible orientations of the triangular configurations of three vortices, so along with the phase portrait on T , we must include another copy of T , which we denote as T^* , having the opposite orientation. More precisely, the integral curves on T^* are precisely those of T except they have the opposite orientation with respect to t . By gluing T and T^* (and their vector fields) smoothly together along their corresponding sides to obtain the double $M := D(T)$ of T , which is a 2-sphere, we obtain a vector field X on M by joining the oriented field corresponding to (13) or (15). Note that at this point, we can reformulate the bundle structure described above for \mathcal{D} in the ‘doubled’ form $\rho : \mathcal{D} \rightarrow M = D(T)$. As for fixed points of the flow, we always have, in terms of trilinear coordinates, $E := (1/3, 1/3, 1/3)$ on T and its copy E^* on the T^* half of M corresponding to an equilateral configuration, and the common vertices $Q_1 = Q_1^*$, $Q_2 = Q_2^*$ and $Q_3 = Q_3^*$ corresponding to points where a pair of vortices coincide, as well as additional points that we shall describe in what follows.

A local stability analysis at E (or E^*) shows that there are three distinct types of dynamical behavior corresponding to the value of $\mathfrak{K}^{(2)} := \mathfrak{K}_3^{(2)}$: These are the *elliptic* case when $\mathfrak{K}^{(2)} > 0$, the *hyperbolic* case

when $\mathfrak{R}^{(2)} < 0$, and the exceptional *parabolic* case when $\mathfrak{R}^{(2)} = 0$. When the system is elliptic, E (and E^*) is a center; E (and E^*) is a saddle point in the hyperbolic case; and when the system is parabolic, E (and E^*) is no longer isolated - it lies on the curve of fixed points C on T (with copy C^* on T^*) defined as

$$C : \kappa_1^{-1}x_1^2 + \kappa_2^{-1}x_2^2 + \kappa_3^{-1}x_3^2 = 0.$$

One of the most remarkable and useful features of the trilinear coordinate based projection of the system (8) on M is what amounts to essentially a unique path lifting property (cf. [51]) for integral curves on M , except along C for parabolic systems. For future reference, we shall refer to this property as the *unique integral path lifting principle* for the bundle $\rho : \mathcal{D} \rightarrow M = D(T)$. More precisely, the initial points in \mathcal{D} for integral curves of (8) establishes a bijective correspondence via the projection ρ with the integral curves of (12) or (14) on M , except along C in the parabolic case. This bijective correspondence can be readily established using the integrals of motion of the systems. As for these flow invariants, employing the constants of motion H_0 and K for (4), it is easy to verify that

$$\kappa_1^{-1} \log R_1 + \kappa_2^{-1} \log R_2 + \kappa_3^{-1} \log R_3 \text{ and } \kappa_1^{-1} R_1^2 + \kappa_2^{-1} R_2^2 + \kappa_3^{-1} R_3^2 \quad (17)$$

are first integrals for (8). These integrals were cleverly combined by Tavantzis & Ting [53] to obtain the following invariants for (13):

$$I := \left(\sum_{k=1}^3 \kappa_k^{-1} x_k^2 \right) \left(\prod_{k=1}^3 x_k^{2/\kappa_k} \right)^{(\kappa_1 \kappa_2 \kappa_3 / \mathfrak{R}^{(2)})}, \quad (18)$$

when $\mathfrak{R}^{(2)} \neq 0$, and

$$I_0 := (x_1/x_3)^{2\kappa_2} (x_2/x_3)^{2\kappa_1} \quad (19)$$

for the parabolic case. The above integrals can be combined (see [12] and [56]) to produce an invariant valid for all cases, namely

$$\bar{I} := \left(\sum_{k=1}^3 \kappa_k^{-1} x_k^2 \right)^{\mathfrak{R}^{(2)}/(2\kappa_3)} \left(x_1^{1/\kappa_1} x_2^{1/\kappa_2} x_3^{1/\kappa_3} \right)^{\kappa_1 \kappa_2}. \quad (20)$$

We note that we shall find the invariant \bar{I} to be particularly useful for our numerical simulations of perturbations of three vortex dynamics in the sequel. In particular, the extent to which \bar{I} differs from a constant value for a perturbed system is a good indication of the extent to which the dynamics of the systems diverges from the LA-integrable motion of the three vortex problem. Any of the above constants of motion can be used to prove the almost unique path lifting property (cf. Synge [52]).

Before embarking on a condensed - but reasonably complete - trilinear coordinate based description of the dynamics in the elliptic, hyperbolic and parabolic cases, certain aspects of the compelling effectiveness of the trilinear approach almost demand comment. We first observe that (8) - and so also (13) - incorporates the Lie group of symmetries of the original system (1) - which is the planar Euclidean group. As one would expect to be able to use the symmetries in a standard symplectic fashion via the associated momentum map to obtain a Hamiltonian restriction of the original system (4) having just one degree of freedom for the three vortex problem, there is the obvious suggestion of a strong link between the symplectic reduction and the system (13). It appears that such a link, which shall have no need of in the sequel, has yet to be definitively established.

Elliptic case ($\mathfrak{R}^{(2)} > 0$)

As indicated above, in the typical elliptic case, which is illustrated for T in Fig. 1, both the stationary point E and its (opposite orientation) copy E^* are centers with index $+1$. It can be easily shown that the fixed points Q_1 , Q_2 and Q_3 are also centers. In addition, there are three other fixed points of (13) - one on the interior of each side of T - denoted as Q_4 , Q_5 and Q_6 in Fig. 1, and each of these additional stationary points can readily be shown to be a saddle points of index -1 . Notice that these observations are consistent with the Poincaré-Hopf Index theorem (see e.g. [31]), inasmuch as a simple calculation yields

$$5(+1) + 3(-1) = 2 = \chi(M) = \chi(D(T)),$$

where $\chi(M)$ is the Euler-Poincaré characteristic of the sphere M , which is two. In this case, we see that the global phase portrait of (15) is completely determined, and this leads to an exhaustive determination of three vortex dynamics in the elliptic case owing to the unique integral path lifting principle mentioned above.

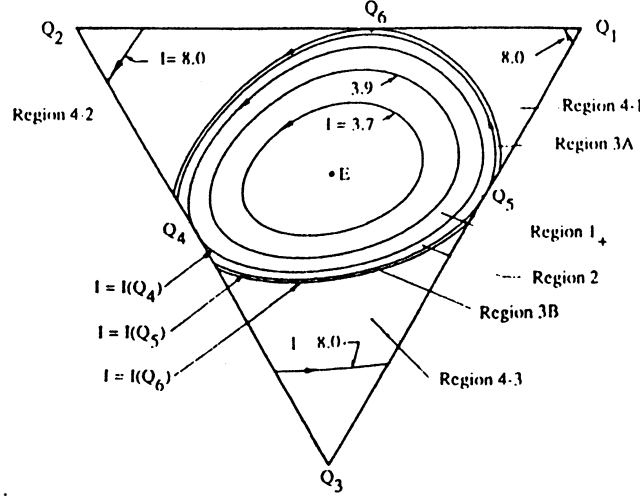


Figure 1: Elliptic type with $\kappa_1 = 2$, $\kappa_2 = 1$, and $\kappa_3 = 1/2$.

Hyperbolic case ($\mathfrak{K}^{(2)} < 0$)

An example of the dynamics for the hyperbolic case is shown in Fig. 2. For all hyperbolic cases, the points E and E^* associated to equilateral configurations are stationary saddle points of index -1 . When it comes to the fixed points at the vertices of the triangle T (glued to those of T^*), the point Q_3 is always a center of index $+1$ for all possible admissible vortex strengths. Any other fixed points, which of course must include the other vertices of T as well other points on the open edges of the triangle can have a variety of natures depending on certain algebraic criteria (see [53]). For example, in the case shown in Fig. 2, Q_1 and Q_2 are both centers, while the stationary points Q_4 , Q_5 and Q_6 are, respectively a center, center and saddle point. Thus we compute the following index sum

$$\text{ind}E + \text{ind}E^* + \text{ind}Q_3 + \text{ind}Q_1 + \text{ind}Q_2 + \text{ind}Q_4 + \text{ind}Q_5 + \text{ind}Q_6 = 2(-1) + 5(+1) + (-1) = 2,$$

which is consistent with the Poincaré-Hopf Index theorem.

Depending on the algebraic criteria (involving the vortex strengths), there are exceptional cases where Q_5 merges with Q_1 or Q_4 coincides with Q_2 , or possibly both, in which case one or both of Q_1 and Q_2 can assume the form of a degenerate isolated stationary point of index $+2$. In all cases, the corresponding index sum can be shown to be in agreement with the Poincaré-Hopf formula. Exceptional cases notwithstanding, once again we also have, for any parameter values consistent with the hyperbolic case, a complete characterization of three vortex dynamics owing to the unique integral path lifting principle.

Parabolic case ($\mathfrak{K}^{(2)} = 0$)

Three vortex dynamics represented in the trilinear coordinate phase plane for an example of the parabolic case is illustrated in Fig. 3. Observe, as was discussed above, in this case the points E and E^* lie on the singular curve C in T and its copy C^* on T^* comprised of stationary points of the system (13). Unfortunately, the unique integral path lifting principle does not apply on the double $D(C)$ of C , which is a circle on M to which the points E and E^* both belong, and with this is associated another problem with the singular curve exhibited in Fig. 3; namely, the trajectories of (13) appear to cross this curve, which is not what one expects for a reasonable dynamical system in which standard uniqueness theorems obtain. But the principle does apply on the complement of $D(C)$ on M , making it possible to obtain an almost complete description of three vortex dynamics in the parabolic case. To fill in the gaps, it is necessary only to conduct a more

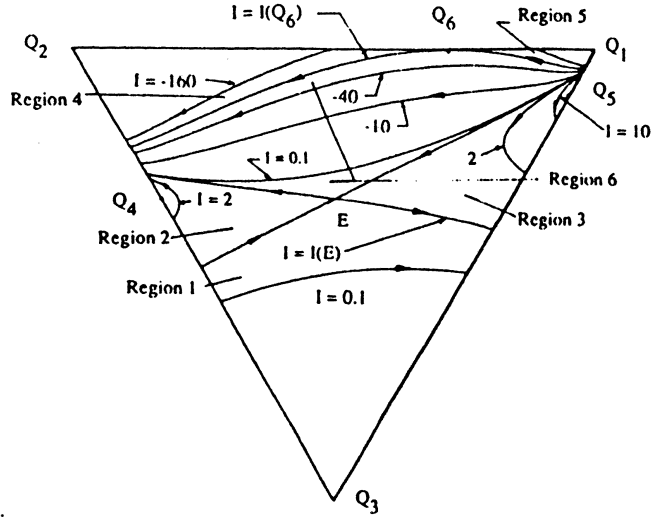


Figure 2: Hyperbolic type with $\kappa_1 = 2$, $\kappa_2 = 1$, and $\kappa_3 = -4/5$.

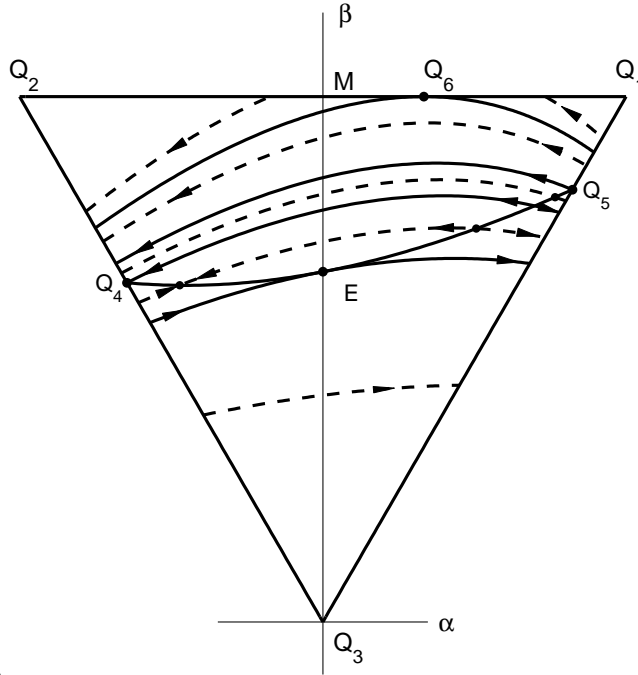


Figure 3: Parabolic type with $\kappa_1 = 2$, $\kappa_2 = 1$, and $\kappa_3 = -2/3$.

careful analysis of the dynamics of the full system in a neighborhood of the singular circle $D(C)$ as in Ting *et al.* [56], but we shall not pursue this further here since it not needed for our analysis in the sequel.

Note that if we cut the sphere along the singular curve $D(C)$ in the example shown in Fig. 3, we obtain two disks B_1 and B_2 for which we can apply the extension of the Poincaré-Hopf formula to manifolds with boundaries. The upper disk B_1 contains the stationary points Q_1 , Q_2 and Q_6 in its interior, and it is easy to verify that these fixed points are, respectively, a center, center, and saddle point. Whence, we find that the index sum in B_1 is

$$2(+1) + (-1) = 1 = \chi(B_1).$$

For the lower hemisphere B_2 , we find that Q_3 is the only interior fixed point, and it is a center of index $+1$. Accordingly we compute that, as expected,

$$1(+1) = 1 = \chi(B_2).$$

Depending on the parameter values of the system for a particular example of the parabolic case, certain dynamic properties can vary from the example depicted in Fig. 3. However, as shown in Tavantzis & Ting [53] (and can also be inferred from the Poincaré-Hopf formula in virtue of the fact that Q_3 is the only stationary point below the singular curve C), the fixed point Q_3 is always a center.

Before leaving our summary of the trilinear coordinate phase plane behavior of the three vortex problem, we wish to emphasize the fact that regardless of the type, the stationary point Q_3 is always a center, and there are often additional centers.

III. TYPES OF PERTURBATIONS

Here we shall provide examples of perturbations three point dynamics that satisfy the hypotheses of the main results that we shall derive in succeeding sections. These particular types of perturbations will also be used for numerical illustrations of where the theorems apply, and where they begin to break down as evidenced by the onset of nonregular, chaotic regimes. The types of perturbations that we choose are three vortex dynamics in a half-plane, restricted four vortex dynamics, and three coaxial vortex ring dynamics in space.

A. Three vortex dynamics in a half-plane

The complex Hamiltonian governing equations for three point vortices of nonzero strengths Γ_k at respective points $z_k = x_k + iy_k$, $1 \leq k \leq 3$, in motion in an ideal fluid in the half-plane $\mathbb{H} := \{z = x + iy \in \mathbb{C} : y > 0\}$ are

$$\kappa_k \dot{z}_k = i \sum_{j=1, j \neq k}^3 \kappa_j \kappa_k (z_j - z_k)^{-1} - i \sum_{j=1}^3 \kappa_j \kappa_k (\bar{z}_j - z_k)^{-1} = 2i \partial_{z_k} H_A, \quad (1 \leq k \leq 3) \quad (21)$$

where the Hamiltonian function is

$$H_A := - \sum_{1 \leq j < k \leq 3} \kappa_j \kappa_k \log |z_j - z_k| + \sum_{1 \leq j \leq k \leq 3} \kappa_j \kappa_k \log |\bar{z}_j - z_k|. \quad (22)$$

With the understanding, expressed in Section II, that smoothness requirements actually demand that the domain be restricted to points in $\mathbb{H}_{\#}^3 := \{\mathbf{z} \in \mathbb{H}^3 : z_j \neq z_k \forall j \neq k\}$, we can express (21) in smooth, real Hamiltonian form

$$\dot{x}_k = \kappa_k^{-1} \partial_{y_k} H_A = \{H_A, x_k\}, \quad \dot{y}_k = -\kappa_k^{-1} \partial_{x_k} H_A = \{H_A, y_k\}, \quad (1 \leq k \leq 3) \quad (23)$$

where we use the same Poisson bracket as in (4). This system, as is easily verified, has the following independent constants of motion in involution

$$H_A, \mathfrak{J}J := \kappa_1 y_1 + \kappa_2 y_2 + \kappa_3 y_3, \quad (24)$$

and this appears to be the maximal such set of invariants: Although we are unaware of a proof showing that (21), or certain analogs for vortex dynamics on a portion of a sphere such as in Kidambi & Newton [35], is in general not LA-integrable, careful numerical studies such as that of Knio *et al.* [36] provide compelling evidence of the existence of chaos for the three vortex problem in the half-plane.

From a perturbation perspective, we can obviously write the Hamiltonian function in the form

$$H_A = H_0 + H_{A,1}, \quad (25)$$

where

$$H_{A,1} := \sum_{1 \leq j \leq k \leq 3} \kappa_j \kappa_k \log |\bar{z}_j - z_k|. \quad (26)$$

Now it is obvious from the nature of these various functions that there exist extensive regions in $\mathbb{H}_\#$ at any positive distance away from the boundary (the x -axis) where we have both $|H_a| \ll |H_0|$ and $|\partial_z H_a| \ll |\partial_z H_0|$, in keeping with our point of view of treating (21) as a (small) perturbation of three vortex dynamics.

B. Restricted four vortex dynamics

By the *restricted four vortex problem (dynamics)*, we mean the motion of four point vortices in an ideal fluid in the (complex) plane \mathbb{C} ($= \mathbb{R}^2$), where three of the vortices, located at points z_1, z_2 and z_3 , have respective nonzero strengths Γ_1, Γ_2 and Γ_3 , while the fourth vortex at the point z_4 has strength Γ_4 satisfying $|\Gamma_4| \ll \mathfrak{m} := \min\{|\Gamma_1|, |\Gamma_2|, |\Gamma_3|\}$ and we have the freedom of making $|\Gamma_4|/\mathfrak{m}$ as small as we wish. It follows from (4) that the complex Hamiltonian governing equations are

$$\kappa_k \dot{\bar{z}}_k = i \sum_{j=1, j \neq k}^4 \kappa_j \kappa_k (z_j - z_k)^{-1} = 2i \partial_{z_k} H_B, \quad (1 \leq k \leq 4) \quad (27)$$

where the Hamiltonian function is

$$H_B := - \sum_{1 \leq j < k \leq 4} \kappa_j \kappa_k \log |z_j - z_k|. \quad (28)$$

Of course this can be put into the following real Hamiltonian form using the same Poisson brackets as in (4):

$$\dot{x}_k = \kappa_k^{-1} \partial_{y_k} H_B = \{H_B, x_k\}, \quad \dot{y}_k = -\kappa_k^{-1} \partial_{x_k} H_B = \{H_B, y_k\}, \quad (1 \leq k \leq 4) \quad (29)$$

where analogous adjustments in the domain are obviously required to insure smoothness. Just as in the case of (4), symmetry considerations show that (26) has the motion invariants

$$H_B, \quad J := \sum_{k=1}^4 \kappa_k z_k, \quad K := \sum_{k=1}^4 \kappa_k |z_k|^2, \quad (30)$$

with the following independent integrals in involution

$$H_B, \quad K, \quad L := (\Re J)^2 + (\Im J)^2. \quad (31)$$

These three independent integrals in involution represent the maximal number, as it has been proved that the system (29) is not integrable in general (see Bagrets & Bagrets [6] and Ziglin [58]). We note, however, that there are certain special cases of four vortex motion on a plane or sphere that are LA-integrable, as shown in Aref & Stremler [3], Borisov *et al.* [17], Eckhardt [23], Sakajo [49], and Sokolovskiy & Verron [50].

Our focus here is to treat (27) or (29) as a perturbation of the three vortex problem. To be mathematically precise, we actually have to consider the system as a perturbation of the three vortex problem embedded in \mathbb{C}^4 , by which we mean

$$\begin{aligned} \kappa_k \dot{\bar{z}}_k &= i \sum_{j=1, j \neq k}^3 \kappa_j \kappa_k (z_j - z_k)^{-1} = 2i \partial_{z_k} H_0, \quad (1 \leq k \leq 3) \\ \dot{\bar{z}}_4 &= 0 = 2i \partial_{z_4} H_0. \end{aligned} \quad (32)$$

Obviously this embedded system is also LA-integrable, and its dynamics consists of a two-parameter infinity of copies of three vortex dynamics - one for each constant value of z_4 . This having been said, there is no harm in the slight abuse of notation that we shall use from now on of treating the restricted four vortex

problem as a perturbation of the three vortex problem. To highlight the perturbation aspect, we write the Hamiltonian function in the form

$$H_B = H_0 + H_{B,1}, \quad (33)$$

where

$$H_{B,1} := -\sigma \sum_{j=1}^3 \kappa_j \log |z_j - z_4|, \quad (34)$$

and we have replaced κ_4 by σ to emphasize the fact that this parameter may be chosen to be as small as necessary to insure the desired perturbation behavior.

C. Three slender coaxial vortex ring dynamics

Consider three (circular) coaxial vortex rings of respective nonzero strengths Γ_1 , Γ_2 and Γ_3 , with axis of symmetry the y -axis, in motion in an ideal fluid in \mathbb{R}^3 . The rings intersect any meridian half-plane bounded along the y -axis in unique points (r_1, y_1) , (r_2, y_2) and (r_3, y_3) , respectively, where r is the distance from the y -axis, and the motion of the rings is completely determined by these points owing to the axial symmetry of the equations of motion. It is convenient to set $z := r^2 + iy = s + iy$, so that the motion of the three intersection points can be considered to be in the half-plane \mathbb{H} . The complex Hamiltonian form of the dynamical equation of motion of these points - de-singularized in a standard way to eliminate the usual infinity in the self-induced velocity of the rings - can be expressed as (*cf.* Blackmore & Knio [8, 9], Blackmore et al. [11], and Lamb [38])

$$\begin{aligned} \kappa_k \dot{z}_k &= \sum_{j=1, j \neq k}^3 \kappa_j \kappa_k r_j r_k (z_j - z_k) \int_0^{\pi/2} \Delta_{jk}^{-3/2} \cos 2\alpha d\alpha - i \kappa_k^2 r_k^{-1} [\log(8r_k \delta_c^{-1}) - 0.558] \\ &\quad - 2i \sum_{j=1, j \neq k}^3 \kappa_j \kappa_k r_j \int_0^{\pi/2} (r_j - r_k \cos 2\alpha) \Delta_{jk}^{-3/2} d\alpha \\ &= 2i \partial_{z_k} H_C, \quad (1 \leq k \leq 3) \end{aligned} \quad (35)$$

where δ_c is a very small positive number representing the common core radii (used in the de-singularization procedure) of the vortex rings,

$$\Delta_{jk} := (r_k - r_j)^2 + (y_k - y_j)^2 + 4r_j r_k \sin^2 \alpha, \quad (36)$$

and the Hamiltonian function is

$$H_C := -2 \sum_{j=1}^3 \kappa_j^2 r_j [\log(8r_j \delta_c^{-1}) - 1.558] - 4 \sum_{1 \leq j < k \leq 3} \kappa_j \kappa_k r_j r_k \int_0^{\pi/2} \Delta_{jk}^{-1/2} \cos 2\alpha d\alpha. \quad (37)$$

This system can, employing the same Poisson bracket used in (4), be recast in the real Hamiltonian form

$$\dot{s}_k = \kappa_k^{-1} \partial_{y_k} H_C = \{H_C, s_k\}, \quad \dot{y}_k = -\kappa_k^{-1} \partial_{s_k} H_C = \{H_C, y_k\}. \quad (1 \leq k \leq 3) \quad (38)$$

It is easy to show that (37) has the two following independent integrals in involution

$$H_C, \quad G := \sum_{j=1}^3 \kappa_j s_j = \sum_{j=1}^3 \kappa_j r_j^2. \quad (39)$$

However, there are no additional independent invariants in involution, as proved in Bagrets & Bagrets [6], so although the dynamics of two coaxial rings is LA-integrable, this is not true in general for the system (35) or (38).

The formulation of (35) as a perturbation of three vortex dynamics is rather more subtle than that of the two preceding examples. Formulas obtained by Callegari & Ting [18] indicate that, relative to a coordinate system moving along the axis of symmetry with the overall translation velocity of the ring configuration, the equations of motion of the rings are closely approximated (modulo a constant factor) by those of three point

vortices at the intersection points of the rings in a meridian plane, when the rings are sufficiently close to one another compared to their distance from the axis of symmetry. More precisely, Blackmore & Knio [8] showed that in a coordinate system moving with the *center of vorticity*

$$z_{cv} := \frac{\kappa_1 z_1 + \kappa_2 z_2 + \kappa_3 z_3}{\kappa_1 + \kappa_2 + \kappa_3} = J/\mathfrak{K}^{(1)}, \quad (40)$$

which is defined for any planar configuration of point vortices as long as $\mathfrak{K}^{(1)} \neq 0$, the equations of motion are Hamiltonian, with a Hamiltonian function of the form

$$\tilde{H}_C = -(\mathfrak{K}_{z_{cv}}/2)[H_0 + H_1], \quad (41)$$

where $|H_1|/|H_0| = o(1/\log \rho)$ and $|\partial_z H_1|/|\partial_z H_0| = o(\rho)$ as $\rho \rightarrow 0$, where ρ is the ratio of the diameter of the configuration of meridian plane points of intersection of the rings to the distance of z_{cv} from the axis of symmetry. As $\mathfrak{K}_{z_{cv}}$ is a constant of motion of (35), the above analysis establishes the three coaxial ring problem as a perturbation of the three vortex problem for rings. We note that the above formulas require that $\mathfrak{K}^{(1)} \neq 0$, and, to simplify matters, we shall usually assume hereafter that this is the case. One can prove the results that we obtain in the sequel without this assumption, but certain rather straightforward modifications are required in our methods of proof in some instances.

IV. QUALITATIVE REGULARITY RESULTS

To get an idea of the type of results that we shall present in this section, we first state a theorem for the examples of the preceding section for the cases in which all of the vortex strengths have the same sign. The proof of this theorem is either contained in, or follows directly from, the results in Blackmore & Knio [8, 9], Blackmore *et al.* [11], Blackmore *et al.* [12], Blackmore & Champanerkar [13], Khanin [32], and Ting & Blackmore [55].

Theorem 1. *Suppose that in each of the perturbation examples in the preceding section the parameters κ_j all have the same sign, so that $K^{(1)} \neq 0$ and the center of vorticity is defined. Then the following properties hold with respect to a moving coordinate system with origin at the center of vorticity:*

- (i) *For the three vortex problem in the half-plane (23) governed by the Hamiltonian function H_A there is a set of initial configurations of positive (Lebesgue) measure with the diameter of the configuration sufficiently small with respect to the distance from the x -axis such that the motion is quasiperiodic on invariant KAM tori, interspersed with periodic orbits.*
- (ii) *The restricted four vortex dynamics governed by H_B according to (29) exhibits, for a set of initial conditions of positive measure, quasiperiodicity on invariant KAM tori along with periodic orbits. This holds when the diameter of the initial configuration is sufficiently small and σ is sufficiently small, or the fourth vortex is sufficiently far from the three larger vortices, or for some combination of both the conditions on σ and the distance of the smaller vortex from the larger vortices.*
- (iii) *Three coaxial vortex ring dynamics generated by H_C in accordance with (38) exhibits the following behavior: There is a set of positive measure of initial configurations with the distances among them sufficiently small compared to their minimum distance to the axis of symmetry that produces quasiperiodic motion on invariant KAM tori and periodic orbits.*

Proof. The proof, as indicated above, follows directly from the cited papers. \square

In the remainder of this section, we shall find rather general conditions on a perturbation term H_1 of a Hamiltonian system

$$\dot{z}_k = \{H, z_k\}, \quad (42)$$

where $1 \leq k \leq 3$, or $1 \leq k \leq 4$ in the embedded version described above for the restricted four vortex problem. Here the Poisson bracket is the same as used in all of the previous equations, the Hamiltonian function is of the form

$$H = H_0 + H_1, \quad (43)$$

and the assumptions are general enough to subsume the persistence of regularity results of Theorem 1, as well as including the same types of perturbations when the vortex strengths associated to the three vortex dynamics generated by H_0 are allowed to differ in sign.

To accommodate all of the perturbations discussed in Section III, we shall consider H to be defined and analytic on an open subset of an appropriate complex unitary space, such as $\mathbb{H}_\#^3$ for the half-plane and coaxial ring problems, and $\mathbb{C}_\#^4$ for the restricted four vortex problem. Moreover, we shall also include the possibility of the perturbation H_1 depending on a real (freely chosen) parameter σ such that

$$|H_1/H_0|, |\partial_{\mathbf{z}} H_1| / |\partial_{\mathbf{z}} H_0| \rightarrow 0 \quad (44)$$

uniformly as $|\sigma| \rightarrow 0$ on any compact subset of the domain on which these expressions are defined. It is convenient to introduce some additional notation. We shall refer to a subset of the domain of definition of (42) as *ample* if it is of positive (Lebesgue) measure in the domain or a submanifold of the domain of complex dimension at least two. Moreover, we shall say that the system (or H) has the *uniform perturbation property* (UPP) if there exists a sequence $\mathcal{S} = \{S_n : n \in \mathbb{N}\}$ of subsets of the domain of the system with nonempty interiors such that $S_{n+1} \subset S_n$,

$$|H_1/H_0| \leq \frac{1}{\log(n+1)}, \text{ and } |\partial_{\mathbf{z}} H_1| / |\partial_{\mathbf{z}} H_0| \leq \frac{1}{n+1} \quad (45)$$

for every $\mathbf{z} \in S_n$ for all $n \in \mathbb{N}$, where \mathbb{N} denotes the positive integers. In this case, we call the sequence \mathcal{S} a *uniform perturbation filtration* (UPF) for (42). We are now in a position to state the main result of this section in a very concise way.

Theorem 2. *Let system (42) satisfy the UPP as described in (45), and also (44) when the perturbation depends on a parameter as in the case of the restricted four vortex problem. Furthermore, suppose that for some UPF, $\mathcal{S} = \{S_n : n \in \mathbb{N}\}$, the system satisfies the property that there is a companion sequence of compact subsets of positive measure $\mathcal{K} = \{K_n : n \in \mathbb{N}\}$ with $K_n \subset S_n$ for all $n \in \mathbb{N}$ such that*

$$\mathbf{z}(0) \in K_{n+1} \implies \mathbf{z}(t) \in S_n \quad (46)$$

for all $t \geq 0$ and for each sufficiently large n , where $\mathbf{z}(t)$ is the solution of (42) initially at $\mathbf{z}(0)$. Then there is an ample set of initial conditions for the system (42), and a set of sufficiently small values of $|\sigma|$, when there is such a free parameter dependence as in the case of the restricted four vortex problem, for which the dynamics includes quasiperiodic motion on an ample set of invariant KAM tori interspersed with a countable collection of periodic orbits.

Proof. Our argument relies heavily on what might be called the limit KAM theorem and a useful generalization of the Poincaré-Birkhoff theorem, which were successfully employed by Blackmore and his collaborators [8, 11, 13] to prove the existence of, respectively, an ample set of invariant tori and periodic orbits for several examples of vortex dynamics problems of the type under consideration.

First, following the same approach as in those papers, we rewrite the Hamiltonian in terms of the action-angle coordinates associated to the unperturbed LA-integrable system as

$$\mathcal{H}(\Lambda, \Theta) = \mathcal{H}_0(\Lambda) + \mathcal{H}_1(\Lambda, \Theta), \quad (47)$$

where the action and angle vectors are, respectively, $\Lambda = (\Lambda_1, \dots, \Lambda_k)$ and $\Theta = (\Theta_1, \dots, \Theta_k)$, with $2 \leq k \leq 4$. We have used slightly different notation for the various Hamiltonians to underscore the fact that it may be necessary, as in the case of coaxial vortex rings, to reduce the number of degrees of freedom by one in order to insure the nondegeneracy of \mathcal{H}_0 , which naturally alters the forms of the original terms of the Hamiltonian (43). This also explains why we have indicated that the number of degrees of freedom k in (47) can assume values between two and four. More specifically, $k = 2$ for the coaxial vortex ring problem, $k = 3$ for three vortex in a half-plane dynamics, and $k = 4$ for the restricted four vortex problem (in which case we recall that the three vortex problem must be embedded in \mathbb{C}^4).

For our purposes it is not necessary to know the exact form of $\mathcal{H}_0(\Lambda)$, which is very difficult to deduce in general; we need only take note of the following readily verifiable properties that hold in any compact subset of the domain in which (42) is analytic:

$$\Delta_1 := \det \left(\frac{\partial^2 \mathcal{H}_0(\Lambda)}{\partial \Lambda_i \partial \Lambda_j} \right) \neq 0, \quad (48)$$

$$\Delta_2 := \det \begin{pmatrix} \frac{\partial^2 \mathcal{H}_0(\Lambda)}{\partial \Lambda_i \partial \Lambda_j} & \frac{\partial \mathcal{H}_0(\Lambda)}{\partial \Lambda_i} \\ \frac{\partial \mathcal{H}_0(\Lambda)}{\partial \Lambda_j} & 0 \end{pmatrix} \neq 0, \quad (49)$$

and

$$\mathcal{H}_0(\Lambda) / \Delta_1, \Delta_1 / \Delta_2 \rightarrow 0 \quad (50)$$

as $|\Lambda|^2 := \Lambda_1^2 + \dots + \Lambda_k^2 \rightarrow 0$ (cf. [8]).

Translating the hypotheses to the transformed Hamiltonian system generated by $\mathcal{H}(\Lambda, \Theta)$; namely

$$\dot{\Lambda}_j = \partial_{\Theta_j} \mathcal{H} = \partial_{\Theta_j} \mathcal{H}_1, \quad \dot{\Theta}_j = -\partial_{\Lambda_j} \mathcal{H} = -\partial_{\Lambda_j} \mathcal{H}_0 - \partial_{\Lambda_j} \mathcal{H}_1, \quad (51)$$

we infer the existence for any $\epsilon > 0$ of a compact set K_ϵ of positive measure of initial conditions for (51) such that

$$|\mathcal{H}_1(\Lambda(t), \Theta(t)) / \mathcal{H}_0(\Lambda(t))|, |\partial_{(\Lambda, \Theta)} \mathcal{H}_1(\Lambda(t), \Theta(t))| / |\partial_{\Lambda} \mathcal{H}_0(\Lambda(t))| \leq \epsilon \quad (52)$$

for all $t \geq 0$. Consequently, owing to the nondegeneracy condition (48), together with (50) in the limiting case for $|\Lambda| \rightarrow 0$, it follows from the KAM theorem that there exists an ample set of invariant (real) k -dimensional tori for (42). If $k = 2$, the isoenergetic nondegeneracy condition (49), coupled with (50) when $|\Lambda| \rightarrow 0$, is enough to insure the existence of periodic orbits. For $k > 2$, the existence of cycles for (42) can be proven, for example, by using the generalization of the Poincaré-Birkhoff fixed point theorem employed by Blackmore *et al.* [11]. Thus, the proof is complete. \square

It is a relatively straightforward task to demonstrate that the three types of perturbations described in the preceding section satisfy the above hypotheses, thereby demonstrating the existence of ample regular dynamics regimes (like those for LA-integrable systems) in each of those cases; to wit, the following result is a direct consequence of Theorem 2.

Corollary 1. *The three vortex in a half-plane system (21), which we denote as (A), the restricted four vortex problem (27), identified as (B), and the three coaxial vortex ring system (35), denoted as (C), all satisfy the hypotheses of Theorem 2, so each of these examples exhibits ample sets of initial configurations leading to quasiperiodic flows on invariant KAM tori accompanied by periodic orbits. More specifically, admissible UPF's of initial conditions leading to regular motion for each of these examples can be described as follows for the elliptic (ET), hyperbolic (HT), and parabolic (PT) types of the unperturbed system:*

(ET) *For (A) there are four kinds of UPF's, denoted as $\mathcal{S}_A^{(0)}$, $\mathcal{S}_A^{(3)}$, $\mathcal{S}_A^{(2)}$ and $\mathcal{S}_A^{(1)}$, with the following characterizations: $\mathcal{S}_A^{(0)}$ is comprised of approximately equilateral configurations (associated to the point E or E^* described in Section II) of the three vortices with diameter sufficiently small compared with their distance from the x -axis; $\mathcal{S}_A^{(3)}$ consists of configurations with $|z_1 - z_2| \ll |z_3 - z_2| \simeq |z_3 - z_1|$ (associated with Q_3 defined in Section II) of diameter sufficiently small compared with the distance of the configuration from the x -axis; $\mathcal{S}_A^{(2)}$, associated with Q_2 , is characterized by making the obvious changes in the definition of $\mathcal{S}_A^{(3)}$; and $\mathcal{S}_A^{(1)}$, corresponding to Q_1 , is defined by making the evident revisions of the description of $\mathcal{S}_A^{(3)}$. There are also four kinds of UPF's for (B), $\mathcal{S}_B^{(0)}$, $\mathcal{S}_B^{(3)}$, $\mathcal{S}_B^{(2)}$ and $\mathcal{S}_B^{(1)}$, defined analogously to $\mathcal{S}_B^{(*)}$, $\mathcal{S}_A^{(3)}$, $\mathcal{S}_A^{(2)}$ and $\mathcal{S}_A^{(1)}$, respectively, wherein the ratio of the diameter of the configuration of the main three vortices to the distance from the x -axis is replaced by the ratio of the diameter to the distance of the configuration from the fourth small vortex; and four kinds for (C), $\mathcal{S}_C^{(*)}$, $\mathcal{S}_C^{(3)}$, $\mathcal{S}_C^{(2)}$ and $\mathcal{S}_C^{(1)}$, also defined analogously to $\mathcal{S}_A^{(*)}$, $\mathcal{S}_A^{(3)}$, $\mathcal{S}_A^{(2)}$ and $\mathcal{S}_A^{(1)}$, respectively, where the point vortices are replaced by the intersection points of the rings with any given meridian plane, and the distance from the axis of symmetry replaces the distance from the x -axis.*

(HT) *There is at least one kind of UPF for (A) corresponding to $\mathcal{S}_A^{(3)}$ as described above, and as many as four others associated with $\mathcal{S}_A^{(2)}$ and $\mathcal{S}_A^{(1)}$, and possibly two others corresponding to the points Q_4 and Q_5 when they are centers, as discussed in Section II. For (B) and (C) there are analogous UPF's that always include those associated to $\mathcal{S}_B^{(3)}$ and $\mathcal{S}_C^{(3)}$, respectively.*

(PT) For (A), (B) and (C) there is always one UPF corresponding to $\mathcal{S}_A^{(3)}$, $\mathcal{S}_B^{(3)}$ and $\mathcal{S}_C^{(3)}$, respectively, and possibly additional UPF associated to any other possible centers as described in the trilinear phase plane characterization of the dynamics covered in Section II.

Proof. We shall provide detailed arguments only for the three vortex system in the half-plane, either when the strengths of all vortices have the same sign (the elliptic type), or they differ in sign (which includes both the hyperbolic and parabolic types). The verifications for the restricted four vortex, and three coaxial vortex ring problems for the various types of the unperturbed (three vortex) system can be obtained analogously by straightforward - but rather lengthy - calculations, and we shall leave the details to the reader, noting that the desired results when the unperturbed system is of elliptic type actually follow directly from Theorem 1.

Even though, as mentioned above, the conclusions we are seeking are a direct consequence of Theorem 1 when the unperturbed system is of elliptic type, we shall include a proof here, since it will be helpful in pointing the way to establishing sufficient conditions for the hyperbolic and parabolic types. Recalling that there is no loss of generality in assuming that $\kappa_1 \geq \kappa_2 \geq \kappa_3 > 0$ when the unperturbed system is elliptic, we first rewrite the constants of motion (24) of (23) in a form more useful to our purposes; namely

$$\begin{aligned} H_A = c_1 &= \kappa_1^2 \log(2y_1) + \kappa_2^2 \log(2y_2) + \kappa_3^2 \log(2y_3) - \kappa_1 \kappa_2 \log \left[\frac{(x_1 - x_2)^2 + (y_1 - y_2)^2}{(x_1 - x_2)^2 + (y_1 + y_2)^2} \right]^{1/2} \\ &\quad - \kappa_1 \kappa_3 \log \left[\frac{(x_1 - x_3)^2 + (y_1 - y_3)^2}{(x_1 - x_3)^2 + (y_1 + y_3)^2} \right]^{1/2} - \kappa_2 \kappa_3 \log \left[\frac{(x_2 - x_3)^2 + (y_2 - y_3)^2}{(x_2 - x_3)^2 + (y_2 + y_3)^2} \right]^{1/2}, \\ \mathfrak{J}J = c_2 &= \kappa_1 y_1 + \kappa_2 y_2 + \kappa_3 y_3, \end{aligned} \quad (53)$$

and define $M(c_1, c_2)$ to be the set of points in $\mathbb{H}_\#^3$ satisfying the pair of equations (53).

It is straightforward to show directly from the form of these defining equations that given any arbitrarily large and small positive number, respectively, λ and ν , there exist $\lambda_* = \lambda_*(\lambda, \nu) \geq \lambda$ and $0 < \nu_* = \nu_*(\lambda, \nu) < \nu$ such that the following property is satisfied if c_1 and c_2 are chosen to be sufficiently large positive numbers: let the initial positions of the vortices, $z_1(0) = x_1(0) + iy_1(0)$, $z_2(0) = x_2(0) + iy_2(0)$, $z_3(0) = x_3(0) + iy_3(0)$ be approximately in the shape of an equilateral triangle (corresponding to E in Fig.1) of diameter less than ν_* , with $y_1(0), y_2(0), y_3(0) \geq \lambda_*$, and define $M_*(c_1, c_2)$ to be the component of $M(c_1, c_2)$ containing the initial point $\mathbf{z}(0)$. Then for all configurations (z_1, z_2, z_3) in the component $M_*(c_1, c_2)$, the diameter is less than or equal to ν , and $y_1, y_2, y_3 \geq \lambda$. Owing to the connectedness of orbits of (23), it must therefore follow that the diameter of the configuration $\mathbf{z}(t) = (z_1(t), z_2(t), z_3(t))$ is less than or equal to ν , and $y_1(t), y_2(t), y_3(t) \geq \lambda$ for all $t \geq 0$, where $\mathbf{z}(t)$ is the solution of (23) with the specified initial condition. Whence the construction of the desired UPF associated to these sets is a simple matter.

If the unperturbed system is hyperbolic or parabolic, then as indicated in Section II, we may - and do - assume without loss of generality that $\kappa_1 \geq \kappa_2 > 0 > \kappa_3$. This puts a slightly different complexion on the system of equations (53), which must be satisfied by all solutions of (23). However, not so different that we cannot use the same kind of argument as for the elliptic type system (given above) modulo a few rather obvious modifications. In fact, we can take our cue for the necessary adjustments by recalling our discussion of the trilinear phase portraits for the three vortex problem in Section II. It is not difficult to see from a close inspection of (53) for the case when κ_3 is negative - in which we concentrate on those terms containing the negative vortex strength - that we can simply change the initial condition on the vortex configuration to conform to the center Q_3 (see Figs 1 and 2). More precisely, we need only change the description of the initial configuration in the preceding paragraph so that $|z_1(0) - z_2(0)| \ll |z_1(0) - z_3(0)| \simeq |z_2(0) - z_3(0)|$ in order to obtain orbits of (23) staying in a component of $M(c_1, c_2)$ analogous to $M_*(c_1, c_2)$. Just as in the previous paragraph, this leads directly to the desired UPF's for the hyperbolic and parabolic cases, and completes the proof for three vortex dynamics in the half-plane. \square

Before moving on to a study of perturbations of specific periodic orbits of three vortex dynamics, we note that the following generalization of Theorem 2 can be easily proved by using essentially the same arguments as in its proof given above.

Theorem 3. Suppose the system (42) is a perturbation of a general LA-integrable Hamiltonian system generated by H_0 , and that H is defined and analytic on an open subset of \mathbb{C}^m , with $m \geq 1$. In addition,

assume that (42) satisfies the hypotheses of Theorem 2, and that the usual KAM nondegeneracy condition holds for H_0 in an admissible UPF. Then there exists an ample set of initial conditions (and small values of a parameter, if pertinent) such that the system exhibits quasiperiodic flows on an ample collection of invariant tori, together with periodic orbits.

V. PERSISTENCE OF PERIODIC ORBITS

In the preceding section we proved theorems demonstrating that various types of, generally non-integrable, Hamiltonian perturbations of certain LA-integrable systems - including that governing the motion of three point vortices in an ideal fluid in the complex plane - have ample dynamical regimes exhibiting the regularity properties that characterize integrable systems; namely quasiperiodic motion on invariant tori and periodic orbits.

Such qualitative results as those already obtained, which are essentially existence theorems, beg the question of how closely such behaviors of the perturbed system approximate those of the unperturbed LA-integrable system associated to H_0 ? This can be viewed as a more quantitative classical perturbation theory related query about the qualitative entities whose existence has been proven by more modern methods in symplectic dynamics. In this section we provide a partial answer to this question as it relates to periodic orbits, which is embodied in the following result.

Theorem 4. *Let the Hamiltonian system defined by (42) and (43) be a perturbation of three vortex dynamics generated by H_0 with respect to a coordinate system moving with the center of vorticity. Suppose that the closed curve C represents a periodic orbit of the unperturbed three vortex system with Hamiltonian function H_0 . Define the (compact) tubular neighborhood $\mathfrak{T}_\epsilon(C)$ for each positive ϵ as*

$$\mathfrak{T}_\epsilon(C) := \{\mathbf{z} : \Delta(\mathbf{z}, C) \leq \epsilon\},$$

where Δ denotes the usual (unitary) distance function. Also define

$$\lambda_\epsilon(C) = \max\{|H_1/H_0|, |\partial_{\mathbf{z}} H_1| / |\partial_{\mathbf{z}} H_0| : \mathbf{z} \in \mathfrak{T}_\epsilon(C)\}.$$

Then if for a given ϵ small enough to insure that $\mathfrak{T}_\epsilon(C)$ is a smooth ($= C^\infty$) submanifold of the domain in which the system (42) is smooth, the quantity $\lambda_\epsilon(C)$ is sufficiently small, (42) has a periodic orbit \tilde{C} in $\mathfrak{T}_\epsilon(C)$.

Proof. First select a point $\mathbf{z}_0 \in C$, and let \mathcal{E}_0 and \mathcal{E} be the energy hypersurface, respectively, of the unperturbed system generated by H_0 and the perturbed system generated by H , which contain the point \mathbf{z}_0 . Both \mathcal{E}_0 and \mathcal{E} have a common odd real dimension, which we denote as $2m - 1$. Let Δ_r denote the Riemannian metric in E induced by the metric Δ , and define

$$B_\delta := \{\mathbf{z} \in \mathcal{E} : \Delta_r(\mathbf{z}, \mathbf{z}_0) \leq \delta\}$$

for small positive values of δ . This is clearly a $(2m - 1)$ -ball in \mathcal{E} for all $\delta \leq \delta_0$ sufficiently small.

We let φ_t represent the flow generated by (42). The following properties follow directly from standard results on differential equations (see e.g. [31]) when $\lambda_\epsilon(C)$ is chosen to be sufficiently small: There exist a transversal Σ through \mathbf{z}_0 for the system (42) in \mathcal{E} and $0 < \delta_2 < \delta_1 < \delta_0$ such that (i) each of the sets $\beta_\delta := B_\delta \cap \Sigma$ is a $2(m - 1)$ -ball in \mathcal{E} whenever $0 < \delta \leq \delta_1$, (ii) the orbits of (42) initially on β_{δ_2} all pass through the interior of β_{δ_1} in finite (positive) time generating a Poincaré map $P : \beta_{\delta_2} \rightarrow \beta_{\delta_1}$ with $P(\beta_{\delta_2}) \subset \text{interior}(\beta_{\delta_2})$, (iii) the radial geodesic curves in E emanating from $\varphi_t(\mathbf{z}_0)$ all transversely intersect the boundary of $\varphi_t(\beta_{\delta_2})$ in unique points for all $0 \leq t \leq t_m$, where t_m is the maximum first return time to Σ among all those for the flow of β_{δ_1} , and (iv) the flow of β_{δ_1} generated by (42) through the first return time to Σ is contained in $\mathfrak{T}_\epsilon(C)$.

Our proof is obviously complete if P has a fixed point, so assume on the contrary that this is not the case. As β_{δ_1} and β_{δ_2} are both (real) odd-dimensional balls, it follows from properties (i)-(iv) that we can modify (42) in the flow of $\beta_{\delta_1} \setminus \beta_{\delta_2}$ in order to obtain an extension \hat{P} of P , such that \hat{P} maps β_{δ_1} into itself and has no fixed points in $\beta_{\delta_1} \setminus \beta_{\delta_2}$. Thus, this smooth self-mapping \hat{P} of β_{δ_1} has no fixed points at all, which contradicts the Brouwer fixed point theorem. Accordingly we conclude that P must itself have a fixed point, which corresponds to a cycle \tilde{C} of (42) in $\mathfrak{T}_\epsilon(C)$, so the proof is complete. \square

Considering the generality of the above result, and the relative simplicity of the proof, it seems as though this should have certainly been discovered. However, the authors were unable to find such a result in the literature, although it appears that it could be derived rather directly from certain results that prove the existence of periodic solutions for Hamiltonian systems employing index theory (*cf.* Blackmore & Wang [10], Golé [26] and Josellis [30]). It is interesting to take note of the very special case where $\partial_{\mathbf{z}}H_1$ in the above theorem takes the form $\partial_{\mathbf{z}}H_1 = \phi(\mathbf{z})\partial_{\mathbf{z}}H_0$, where ϕ is a smooth, real valued function whose magnitude can be made arbitrarily small in the tubular neighborhood $\mathfrak{T}_\epsilon(C)$. Then as one can readily see by making the obvious transformation of the time parameter, the cycle of the perturbed system is not just an approximation of C , it is identical with it, and the local flow of the perturbed system is identical with that of the unperturbed system (modulo a change of parametrization).

Theorem 4 can be applied directly to the perturbations of three vortex dynamics that we have been considering to identify conditions under which the perturbation has a periodic orbit close to one for the three vortex problem. We leave the very straightforward proof of the next result to the reader.

Corollary 2. *The hypothesis and conclusions of Theorem 4 regarding the existence of a periodic orbit of the perturbed system close to a periodic orbit $C : z = z(t)$, $0 \leq t \leq p$, of the unperturbed (three vortex) system hold for (A) three vortices in a half-plane, (B) restricted four vortex dynamics, and (C) three slender coaxial ring dynamics obtain under the following conditions:*

- (a) *Each of the coordinates of $z(t)$ on C is sufficiently distant from the x -axis in the complex plane.*
- (b) *For the coordinates $z(t) = (z_1(t), z_2(t), z_3(t), z_4(t)) \in C$, the distance in the complex plane \mathbb{C} from $z_4(0)$ to the set $\{z \in \mathbb{C} : z = z_1(t), z_2(t) \text{ or } z_3(t), 0 \leq t \leq p\}$ is sufficiently large, or the magnitude of the strength of the fourth vortex is sufficiently small, or a suitable combination of both of these conditions is enforced.*
- (c) *The coordinates of $z(t)$ on C representing the points of intersection of the rings with a meridian plane are sufficiently far from the axis of symmetry (represented by the y -axis in this plane).*

We note that the conditions given in Corollary 2 by no means exhaust all possible situations where the perturbed system has a periodic orbit close to a periodic orbit of the unperturbed system. For example, it is easy to see that in the restricted four vortex problem there are many such cases where the fourth small vortex is neither particularly small nor very distant from the three large vortices: Simply consider any configuration of the large vortices that generates a periodic solution of the three vortex problem, and place the fourth vortex, of any strength, at the center of vorticity of the larger vortices. Then the fourth vortex remains fixed, and the motion of the three larger vortices is unaffected by its presence.

VI. SIMULATIONS OF DYNAMICS

In this section we provide numerical examples that illustrate the persistence of regular motion for different types of perturbations, as well as breakdown of regularity as large perturbations are considered and the hypotheses of our theoretical results are accordingly violated. As mentioned earlier, we consider half-plane dynamics, restricted four vortex perturbations, and slender coaxial vortex ring dynamics, which we refer to below as type A, B, and C, respectively.

Our simulations cover just a small sample of possible cases for the various perturbations, and they are presented in two basic types of graphical forms: As trajectories in the plane or half-plane, or as Poincaré sections, which are especially well suited to illuminating transitions from regular to chaotic motion. In particular, for the unperturbed system we present plots of the trajectories of the three vortices in the plane, juxtaposed with the corresponding trilinear phase plane stationary point. If any of the perturbation types satisfy the hypotheses of Corollaries 1 and 2, we expect that plots of the trajectories of their vortex elements (with respect to a coordinate system moving with the configuration) would show a small variation of the plots for the unperturbed system. Also included for type A perturbations are Poincaré maps, along with the corresponding Poincaré map for the unperturbed system, showing that there is a transition to chaotic motion as the array starts closer to the boundary of the half-plane compared to the diameter of the initial configuration of vortices. For type C perturbations, we present Poincaré maps exhibiting strong regularity when the slender coaxial rings are very close to one another compared to the distance of the configuration

from the axis of symmetry, as expected in view of Corollary 1. For the restricted four vortex problem, a pair of Poincaré maps shows how the motion tends to be more chaotic as the initial position and strength of the fourth small vortex, respectively, starts closer to the initial group of three larger vortices and grows in comparison to the strengths of this group. This behavior is entirely consistent with Corollaries 1 and 2.

For type A and C perturbations, we consider the setup illustrated in Fig. 4. As shown in the figure, the geometry is specified in terms of two parameters, b and a , while the strength of the perturbation is reflected by the height h for type A, or the mean radius ρ for type C. Introducing the parameter $c \equiv \sqrt{a^2 + b^2}$, an analysis was conducted of the nine cases summarized in Table 1.

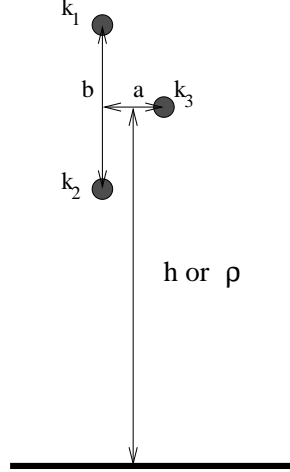


Figure 4: Initial configuration for type I and II perturbations.

| Case | b | c | κ_1 | κ_2 | κ_3 | $\mathfrak{K}^{(2)}$ |
|------|------|-----|------------|------------|------------|----------------------|
| 1 | 0.1 | 1 | 2 | 1 | 0.5 | 3.5 |
| 2 | 0.1 | 1 | 2 | 1 | -2/3 | 0 |
| 3 | 0.1 | 1 | 2 | 1 | -0.8 | -0.4 |
| 4 | 2/7 | 1 | 2 | 1 | 0.5 | 3.5 |
| 5 | 2/7 | 1 | 2 | 1 | -2/3 | 0 |
| 6 | 2/7 | 1 | 2 | 1 | -0.8 | -0.4 |
| 7 | 0.49 | 1 | 2 | 1 | 0.5 | 3.5 |
| 8 | 0.49 | 1 | 2 | 1 | -2/3 | 0 |
| 9 | 0.49 | 1 | 2 | 1 | -0.8 | -0.4 |

Table 1: Summary of inputs. The calculated value of $\mathfrak{K}^{(2)}$ is also reported.

Summary of Simulation Results

- Our main focus is on examples where the unperturbed system is of the elliptic type, *i.e.* cases 1, 4, 7. Initial configuration near Q_3 , in between two separatrices (region 3B shown in Fig. 1), and near the center E . We expect more sensitivity to perturbations for any of the three types for case 4 than 1 and 7, since case 4 represents a configuration closer to the separatrix shown in Fig. 1, and this is borne out by our simulations.
- The motion of the three vortices in the plane for cases 1, 4, and 7 in the unperturbed system is shown in Figure 5. Notice that even in this case, the trajectories are considerably more complicated for case 4 than cases 1 and 7. Again this is due to proximity to the separatrix.

- For a type A perturbation with $h = 10$, we observe little impact on trajectories of the 3 vortices. This is consistent our theoretical results.
- For cases 1 and 7, we also consider lower values of h - as low as $h = 2.5$. For case 4 corresponding, to a point close to the separatrix for the unperturbed system, the dynamics become much more complicated at a more rapid pace as h decreases. The Poincaré maps (x_1, y_1 for $y_2 = 0$) shown in Fig. 6 suggest that when $h = 2.5$ the dynamics is starting to exhibit chaotic regimes.
- Also considered in our simulations are type C perturbations. We found that it is necessary to start with a large mean radius for the three slender coaxial vortex rings in order to observe theoretical predictions. With $\rho = 250$, the motion remains regular. This is illustrated in the Poincaré maps shown in Fig. 7.
- In the final set of simulations, we considered examples of restricted four vortex dynamics The configuration of the three larger (principal) vortices, which we designate as vortices 1-3, is defined in terms of equal-strength vortices, $k_i = 1$ for $i = 1, 2, 3$, initially located on the vertices of an equilateral triangle of side 1. Specifically, the initial conditions are given by: $z_1 = (-0.5, 0)$, $z_2 = (0.5, 0)$, and $z_3 = (0, h)$ where $h = \sqrt{3}/2$. For the unperturbed case, the trajectories with respect to the centroid are identical circles centered at the origin. We consider perturbation due to the introduction of a weak vortex having $k_4 = 0.1$. Five different initial conditions for the fourth vortex are considered, namely $z_4 = (0.1, 0)$, $z_4 = (0.1, h/3)$, $z_4 = (0.1, 2h/3)$, $z_4 = (0.1, h)$, and $z_4 = (0.1, 4h/3)$. We refer to these as cases i - v, respectively.
- Consistent with predictions, the trajectories of vortices 1-3 are only weakly affected by the introduction of the fourth (weaker or smaller) vortex (not shown). More precisely, we find that the common circle of motion of the three principal vortices becomes slightly thickened (forming a thin annulus) in the plane - with the thickness of the annulus and variation of motion within it reflecting the magnitude (as well as the particular form) of the perturbation caused by the smaller vortex. One expects the annulus to thicken when the smaller vortex starts closer to the configuration of vortices 1-3, its strength increases, or when the weaker vortex starts in a position that has a more subtle connection with the configuration of the whole set of four vortices. These subtleties will be discussed at some length in the following points.
- The most dramatic effect on the dynamics for the restricted four vortex problem is naturally manifested in the motion of the fourth (smaller or weaker) vortex. The trajectory of the fourth vortex can be either regular or complex and even chaotic, depending on its initial location and strength. Moreover, the complexity of its trajectory is quite sensitive to changes in its initial location and strength. This is illustrated in the Poincaré maps of (x_4, y_4 at $y_1 = 0$).
- The Poincaré maps for the motion of the fourth vortex shown in Fig. 8 indicate via the characteristic splattering that the dynamics for cases i and ii exhibit typical chaotic behavior, or at least a transition from regular to chaotic motion, whereas cases iii and iv are essentially regular. This may appear to be somewhat surprising in light of the fact that the distance from vortex 4 to the set of vortices 1-3 in cases i and ii is not very different from the distances in cases iii and iv.
- A plausible explanation for the somewhat counter intuitive behavior described above is as follows: If we view vortex 4 as perturbing the motion of vortices 1-3 (see Fig. 1), we would not, as indicated above, expect this to have much effect on the dynamics of vortices 1-3. In particular, the configuration of vortices 1-3 places it initially at the center E , so the inherent stability should preserve periodic motion for this configuration that varies only slightly from an equilateral array. Dually, we can consider the triple of vortices 1, 2 and 4, as being perturbed by the strong third vortex. For cases i and ii, the initial configuration of $\{1, 2, 4\}$ is rather close to a separatrix shown in Fig. 8. This experiences a strong perturbation from vortex 3, which is substantial enough to push the configuration across the separatrix and also possibly break this curve. Similarly, cases iii-v are such that the initial configuration of vortices i, ii and iv are substantially further removed from such a separatrix, so then it is not surprising that the dynamics for cases i and ii is far less regular than that for cases iii and iv.

- Some of these subtle points are further illustrated and contrasted in Fig. 9, which shows the dynamics of the fourth vortex, again using Poincaré maps. The dynamics of the fourth vortex of finite strength, portrayed in the left column of figures, is contrasted with the same cases on the right, where the strength of vortex 4 is taken as zero. In other words, we are treating vortex 4 as a passive particle propelled by the motion of the three stronger vortices. Notice that in all cases where the strength of the vortex is zero, the Poincaré maps show that the motion of this vortex is regular. The juxtaposition in Fig. 9 underscores the point that the effect of the fourth vortex is dependent both on its initial placement and strength.

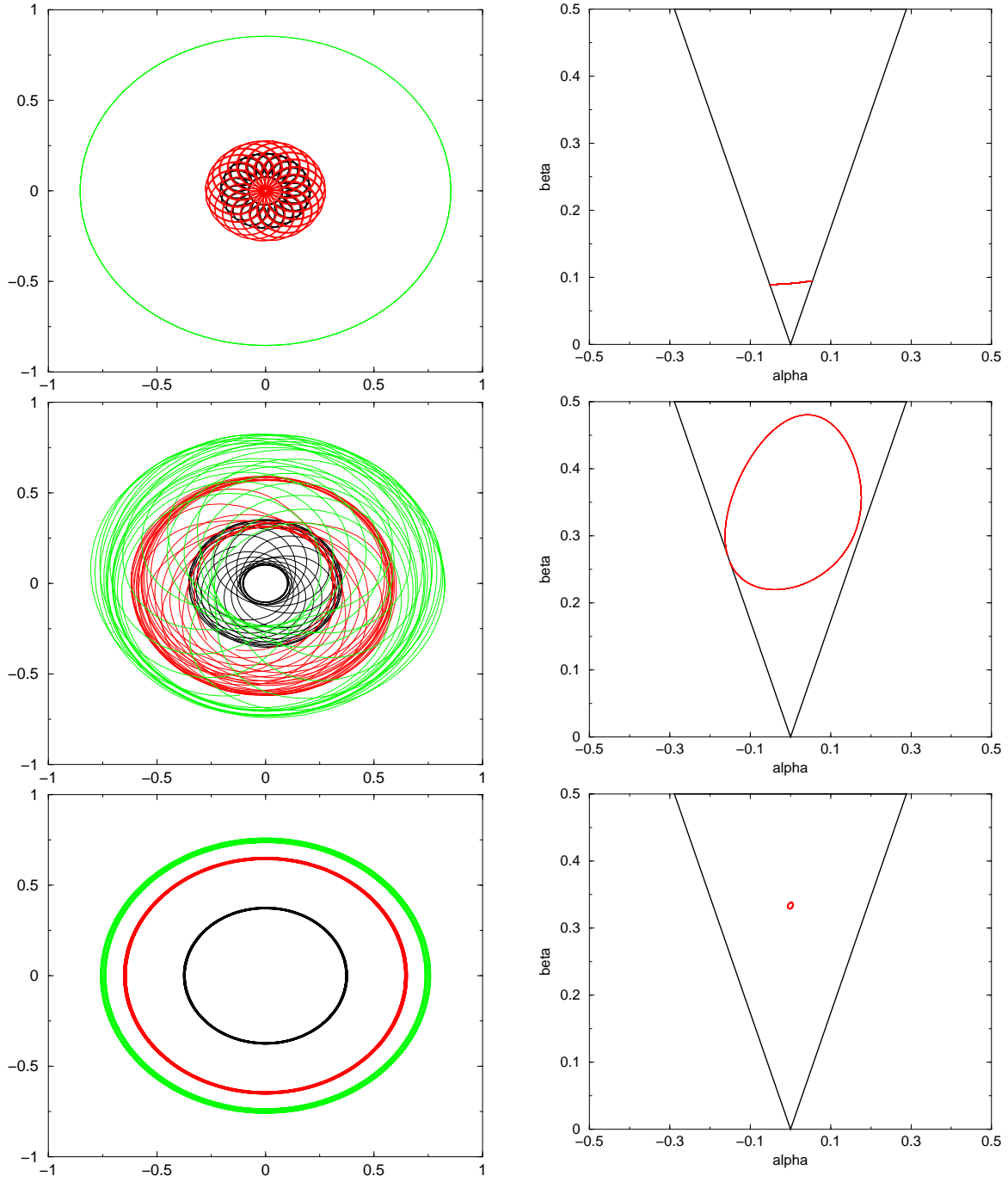


Figure 5: Trajectory of the three vortices with respect to the centroid (left) and in the $\alpha - \beta$ plane (right): case 1 (top), case 4 (middle) and case 7 (bottom). The trajectory of vortex 1 is depicted in black, while those of vortices 2 and 3 are shown in red and green, respectively.

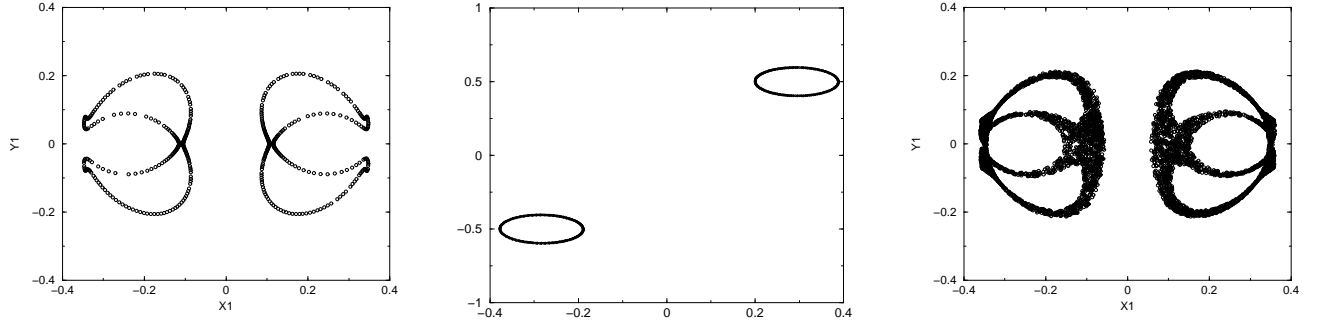


Figure 6: Poincaré maps for case 4. Left: unperturbed system; middle: type A perturbation with $h = 10$; right: type A perturbation with $h = 2.5$.

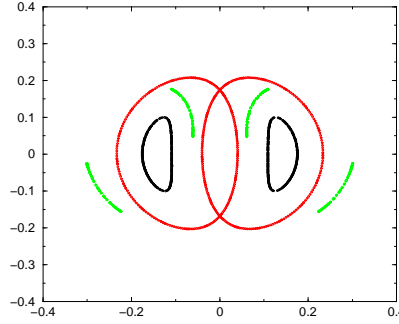


Figure 7: Poincaré maps for case 1 (black), 4 (red), and 7 (green). Type C perturbation with $\rho = 250$.

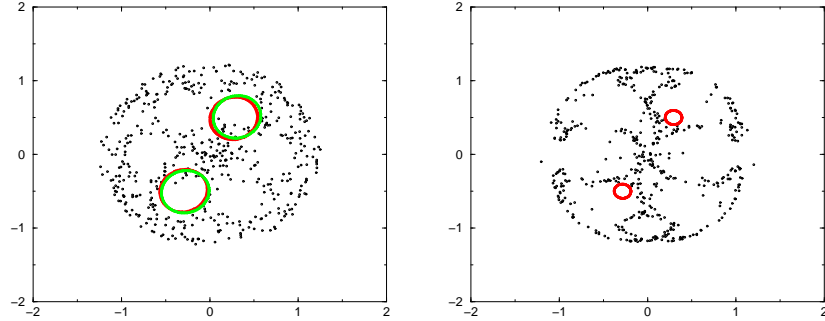


Figure 8: Poincaré maps for the restricted 4-vortex configuration. Left: maps for case i (black), iii (red) and v (green). Right: maps for case ii (black) and iv (green).

VII. CONCLUDING REMARKS

Following a brief summary of the dynamics of three point vortices moving in an ideal fluid in the plane in both a Hamiltonian and trilinear coordinate context, which highlighted the properties most pertinent for our investigation - such as the existence of a center regardless whether the unperturbed system is of elliptic, hyperbolic or parabolic type corresponding, respectively, to $\kappa_1\kappa_2 + \kappa_1\kappa_3 + \kappa_2\kappa_3$ positive, negative or zero (assuming that $\kappa_1 + \kappa_2 + \kappa_3 \neq 0$) - we described the three kinds of perturbations that comprised the focus of this paper. These were three vortices in a half-plane, a restricted four vortex problem, and the motion of three coaxial vortex rings.

We then formulated and proved some new results on the existence of regular regimes - characterized by behavior associated to integrable Hamiltonian systems - for non-integrable perturbations of LA-integrable

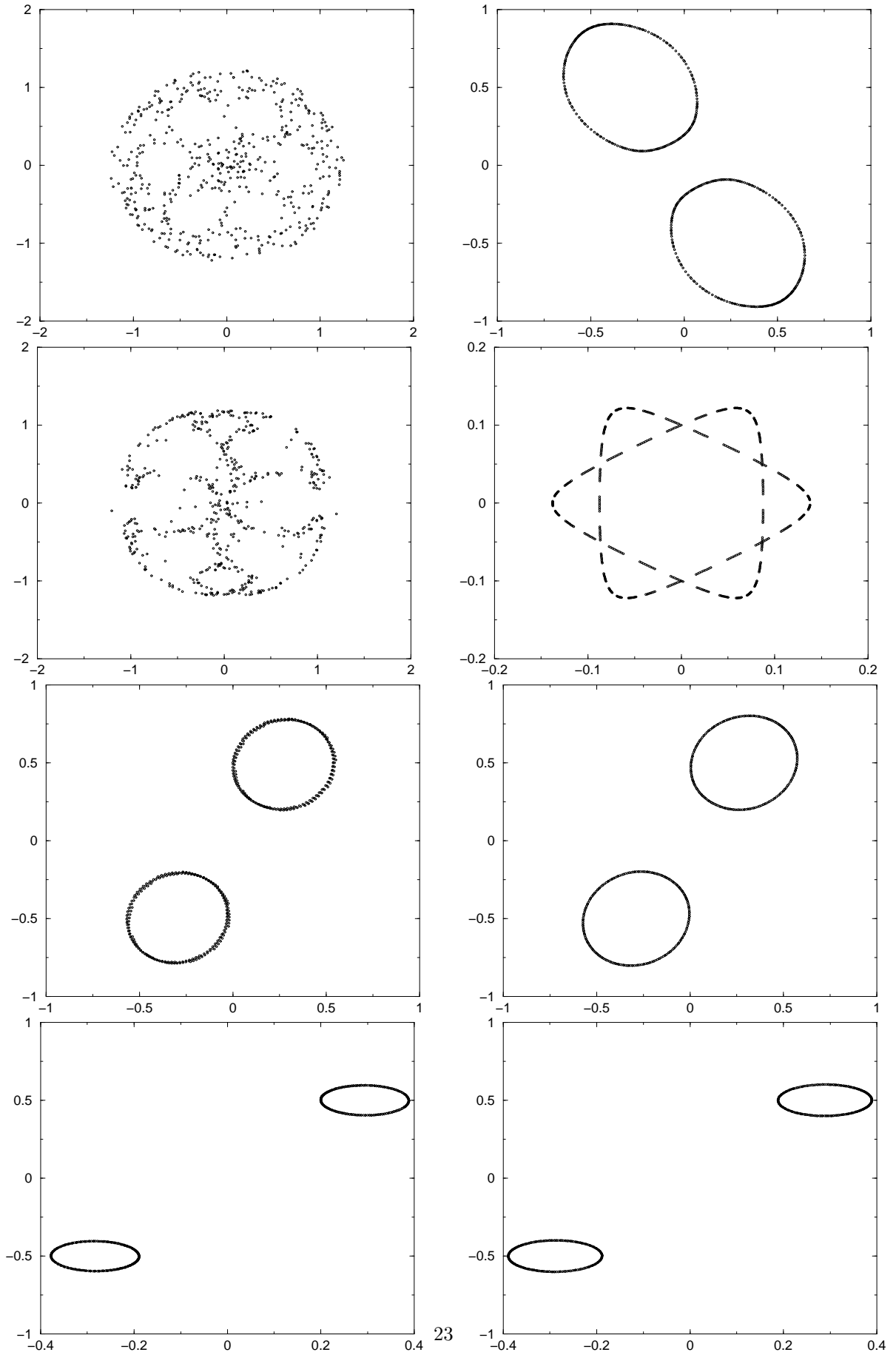


Figure 9: Poincaré maps for the restricted 4-vortex configuration, for perturbed (left) and unperturbed (right) systems. Shown are results for cases i, ii, iii, and iv, arranged from top.

three vortex dynamics. The regular dynamics in point includes the existence of an ample set of quasiperiodic flows on invariant KAM tori along with periodic solutions. These results, which were proved employing rather modern variants and generalizations of the KAM and Poincaré-Birkhoff fixed point theorems, extend some related recent results for point and coaxial ring vortices to cases where the signs of the vortex strengths differ, and are such that their sufficient conditions can be readily checked. But, they are really just existence theorems that provide little information about the location and finer geometric details of the predicted invariant tori and periodic orbits.

The lack of quantitative specificity in our first set of perturbation theorems was subsequently addressed for the case of periodic orbits, with some novel results concerning the existence of periodic solutions for perturbed systems that closely approximate cycles of the unperturbed three vortex system. These results - possessing the look and feel of classical perturbation theorems - were proved using Poincaré sections in concert with the Brouwer fixed point theorem, are stated in terms of easily verifiable criteria guaranteeing the existence of periodic orbits of the perturbed system close to those of the three vortex system.

Several numerical simulations were performed to illustrate the perturbations theorems obtained in a variety of contexts. Also included were examples to indicate how the desired regularity of the dynamics breaks down as the assumptions of our theorems are stretched to the limit, which is accomplished by showing how there is a (transitional) preponderance of chaotic regimes as these limits are exceeded.

Our study of perturbations of the three vortex problem has, we hope, shed some new light on these particular systems as well as more general vortex phenomena. It certainly has - as is often the case in such studies - created more interesting questions than it has answered, several of which we intend to investigate in the near future. For example, our work here suggests that most of our results can be extended to systems comprised of a larger number of vortex elements if they are clustered in certain ways. As an illustration of this, suppose that one has a pair of vortices of strengths of the same sign that are initially quite close together. Then this pair constitutes a binary system that should, under the right conditions, remain close together for all time, and behave as if it were a single vortex of strength equal to the sum of the strengths of each of its components if any other point vortices have initial distances from this pair that are substantially larger than the distance between the pair. A configuration of four vortices including such a binary system and two other vortices initially quite distant from the pair, might well be expected to behave very much like a slightly perturbed three vortex system in the plane (with the binary system acting like a single vortex), thus generating a profusion of regimes exhibiting regular (integrable type) dynamics. Interesting clustering problems of this kind are something that we plan to turn our attention to in our future research.

Another rather manifest question raised by our investigation - that we intend to presently turn our attention to - concerns extending the quantitative perturbation results obtained for periodic orbits to invariant tori. It appears that it may be possible, using some type of generalization of the Poincaré-Birkhoff fixed point theorem, to show that in certain cases the perturbed system has an invariant torus that is close to an invariant torus for the unperturbed system.

ACKNOWLEDGMENTS

The authors wish to dedicate this paper to Professor Theodore Yaotsu Wu of the California Institute of Technology, belatedly on his eightieth birthday, and to honor his life-long pioneering contributions to fluid mechanics and soliton waves, and his leadership in training younger scientists and engineers. In particular, L. T. would like to refresh his memories with Professor Wu of being classmates in the ninth grade at Guang Hua Middle School and 1946 alumni of the Chiao Tung University, Shanghai.

In addition, the authors wish to express their appreciation to Charles Doering and Paul Newton, guest editors of the special focus issue of the Journal of Mathematical Physics on mathematical fluid dynamics, for their gracious invitation to contribute this paper.

Lastly, we wish to acknowledge that preliminary versions of some of the material in this paper were reported on at the ICTAM Conference in Warsaw, Poland, 2004, the Third MIT Conference on Computational Fluid and Solid Mechanics, 2005, and the GAMM Annual Meeting, Berlin, 2006, and appeared in summary form in the proceedings of the first two conferences.

References

- [1] H. Aref, Motion of three vortices, *Phys. Fluids* **22** (1979), 393-400.
- [2] H. Aref, Integrable, chaotic, and turbulent motion in two-dimensional flows, *Ann. Rev. Fluid Mech.* **15** (1983), 345-389.
- [3] H. Aref and M. Stremler, Four-vortex motion with zero total circulation and impulse, *Phys. Fluids* **11** (1999), 3704-3715.
- [4] V. Arnold, Small denominators and the problem of stability of motion in classical and celestial mechanics, *Russian Math. Surveys* **18** (1963), 85-91.
- [5] V. Arnold, *Mathematical Methods of Classical Mechanics*, Springer-Verlag, New York, 1978.
- [6] A. Bagrets and D. Bagrets, Non-integrability of two problems in vortex dynamics, *Chaos* **7** (1997), 368-375.
- [7] G. Birkhoff, An extension of Poincaré's last geometric theorem, *Acta Math.* **47** (1925), 210-231.
- [8] D. Blackmore and O. Knio, KAM theory analysis of the dynamics of three coaxial vortex rings, *Physica D* **140** (2000), 321-348.
- [9] D. Blackmore and O. Knio, Transition from quasiperiodicity to chaos for three coaxial vortex rings, *ZAMM* **80 S** (2000), 173-176.
- [10] D. Blackmore and C. Wang, Morse index for autonomous linear Hamiltonian systems, *Int. J. Diff. Eqs. and Appl.* **7** (2003), 295-309.
- [11] D. Blackmore, J. Champanerkar and C. Wang, A generalized Poincaré-Birkhoff theorem with applications to coaxial vortex ring motion, *Discrete and Contin. Dyn. Syst.-B* **5** (2005), 15-33.
- [12] D. Blackmore, L. Ting and O. Knio, Motion of three point vortices, *Proc. 3rd MIT Conf. on Comp. Fluid and Solid Mech.*, June, 2005.
- [13] D. Blackmore and J. Champanerkar, Periodic and quasiperiodic motion of point vortices, *Vortex Dominated Flows*, D. Blackmore, E. Krause and C. Tung (eds.), World Scientific, Singapore, 2005, pp. 21-42.
- [14] V. Bogomolov, Two-dimensional fluid dynamics on a sphere, *Izv. Atmos. Oc. Phys.* **15** (1979), 18-22.
- [15] A. Borisov and V. Lebedev, Dynamics of three vortices on a plane and a sphere - II, *Regul. Chaotic Dyn.* **3** (1998), 99-114.
- [16] A. Borisov, I. Mamaev and A. Kilin, Absolute and relative choreographies in the problem of point vortices moving on a plane, *Reg. & Chaot. Dyn.* **9** (2004), 101-111. 99-114.
- [17] A. Borisov, I. Mamaev and A. Kilin, New periodic solutions for three or four identical vortices on a plane and sphere. arXiv.nlin.SI/0507058 v1 26 Jul 2005.
- [18] A. Callegari and L. Ting, Motion of a curved vortex filament with decaying vortical core and axial velocity, *SIAM J. Appl. Math.* **35** (1978), 148-175.
- [19] A. Carter, An improvement of the Poincaré-Birkhoff fixed point theorem, *Trans. Amer. Math. Soc.* **269** (1982), 285-299.
- [20] A. Celletti and C. Falcolini, A remark on the KAM theorem applied to a four vortex system, *J. Stat. Phys.* **52** (1988), 471-477.
- [21] C. Conley and E. Zehnder, The Birkhoff-Lewis fixed point theorem and a conjecture of V. I. Arnold, *Invent. Math.* **73** (1983), 207-253.

- [22] W.Y. Ding, A generalization of the Poincaré-Birkhoff theorem, *Proc. Amer. Math. Soc.* **88** (1983), 341-346.
- [23] B. Eckhardt, Integrable four vortex motion, *Phys. Fluids* **31** (1988), 2796-2801.
- [24] A. Floer, Symplectic fixed points and holomorphic spheres, *Comm. Math. Phys.* **120** (1989), 575-611.
- [25] J. Franks, Generalizations of the Poincaré-Birkhoff theorem, *Annals of Math.* **128** (1988), 139-151.
- [26] C. Golé, *Symplectic Twist Maps*, World Scientific, Singapore, 2001.
- [27] D. Goryachev, *On Certain Cases of Motion of Rectilinear Parallel Vortices*, Moscow University Printing House, Moscow, 1898. (in Russian)
- [28] W. Gröbli, *Speciale Probleme über die Bewegung Geradliniger Paralleler Wirbelfäden*, Zürcher and Furrer, Zürich, 1877.
- [29] H. Helmholtz, Über integrale hydrodynamischen gleichungen weiche den wirbelbewegungen entsprechen, *J. Reine Angew. Math.* **55** (1858), 25-225.
- [30] F. Josellis, Lyusternik-Schnirelman theory for flows and periodic orbits for Hamiltonian systems on $T^n \times \mathbb{R}^n$, *Proc. London Math. Soc.* **68** (1994), 641- 672.
- [31] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Cambridge University Press, Cambridge, 1995.
- [32] K. Khanin, Quasi-periodic motion of vortex systems, *Physica D* **4** (1982), 261-269.
- [33] G. Kirchhoff, *Vorlesungen über Mathematische Physik, Vol. 1*, Teubner, Leipzig, 1876.
- [34] R. Kidambi and P. Newton, Motion of three point vortices on a sphere, *Physica D* **116** (1998), 143-175.
- [35] R. Kidambi and P. Newton, Point vortex motion on a sphere with solid boundaries, *Phys. Fluids* **12** (2000), 581-588.
- [36] O. Knio, L. Collerec and Juvé, Numerical study of sound emission by 2D regular and chaotic vortex configurations, *J. Comput. Phys.* **116** (1995), 226-246.
- [37] L. Kuznetsov and G. Zaslavsky, Regular and chaotic advection in the flow field of a three-vortex system, *Phys. Rev. E* **58** (1998), 7330-7349.
- [38] H. Lamb, *Hydrodynamics*, Dover, New York, 1932.
- [39] C.C. Lim, Existence of KAM tori in the phase-space of lattice vortex systems, *ZAMP* **41** (1990), 227-244.
- [40] C.C. Lim, Relative equilibria of symmetric N-body problems on a sphere: inverse and direct results, *Comm. Pure Appl. Math.* **51** (1998), 341-371.
- [41] C.C. Lin, *On the Motion of Vortices in Two Dimensions*, Toronto University Press, Toronto, 1943.
- [42] J. Moser, *Stable and Random Motions in Dynamical Systems*, Princeton Univ. Press, Princeton, 1973.
- [43] J. Moser, Proof of a generalized form of the fixed point theorem due to G. D. Birkhoff, *Geometry and Topology*, Lecture Notes in Math., **597**, Springer-Verlag, New York, 1977.
- [44] P. Newton, *The N-Vortex Problem: Analytical Techniques*, Springer-Verlag, New York, 2001.
- [45] E. Novikov, Dynamics and statistics of a system of vortices, *Sov. Phys. JETP* **41** (1975), 937-943.
- [46] H. Poincaré, Sur un théorème de géométrie, *Rend. del. Circ. Math. du Palermo* **33** (1912), 375-407.
- [47] H. Poincaré, *Théorie des Tourbillons*, G. Carré (ed.), Deslis Frères, Paris, 1983.

- [48] C. Robinson, Generic properties of conservative systems I; II, *Amer. J. Math.* **92** (1970), 562-603;897-906.
- [49] T. Sakajo, Integrable four-vortex motion on sphere with zero moment of vorticity. (preprint)
- [50] M. Sokolovskiy and J. Verron, Four-vortex motion in the two layer approximation: integrable case, *Regul. Chaotic Dyn.* **5** (2000), 413-436.
- [51] E. Spanier, *Algebraic Topology*, McGraw-Hill, New York, 1966.
- [52] J. Synge, On the motion of three vortices, *Can. J. Math.* **1** (1949), 257-270.
- [53] J. Tavantzis and L. Ting, The dynamics of three vortices revisited, *Phys. Fluids* **31** (1988),1392-1409.
- [54] L. Ting and R. Klein, *Viscous Vortical Flows*, Lecture Notes in Physics, Vol. **374**, Springer-Verlag, Berlin, 1991.
- [55] L. Ting and D. Blackmore, Bifurcation of motions of three vortices and applications, *Proc. 2004 ICTAM Conf.*, Warsaw, 2004.
- [56] L. Ting, D. Blackmore and O. Knio, Perturbed three point vortex problems - the parabolic case. (in preparation)
- [57] T. Tokieda, Tourbillons dansants, *C. R. Acad. Sci. Paris Série I* **333** (2001), 943-946.
- [58] S. Ziglin, The non-integrability of the problem of motion of four vortices of finite strengths, *Physica D* **4** (1982), 268-269.

